Transformées de Fourier

Définitions

TF Directe

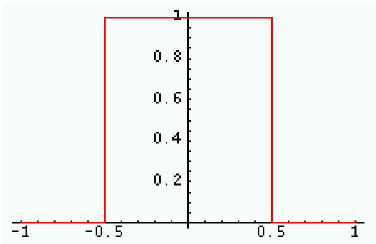
$$\hat{f}(\nu) = \int_{-\infty}^{\infty} f(t)\ e^{-2i\pi\nu t}\ dt$$

TF Inverse

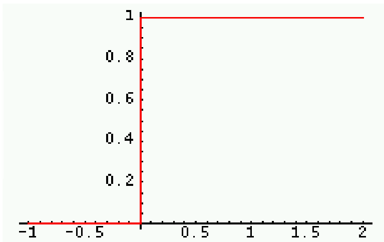
$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\nu)\ e^{2i\pi\nu t}\ d\nu$$

Divers

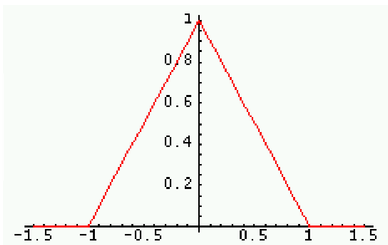
- Fonction porte : $\Pi(t) = \begin{cases} 1 \text{ si } |t| < 1/2 \\ 0 \text{ si } |t| > 1/2 \end{cases}$



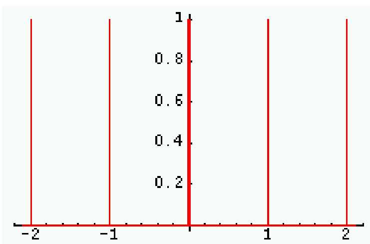
- Fonction de Heaviside : $H(t) = \begin{cases} 1 \text{ si } t > 0 \\ 0 \text{ si } t < 0 \end{cases}$



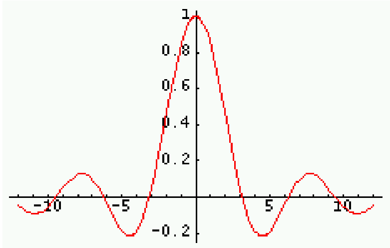
- Fonction triangle : $\Lambda(t) = \begin{cases} 1 - |t| \text{ si } |t| \leq 1 \\ 0 \text{ sinon} \end{cases}$



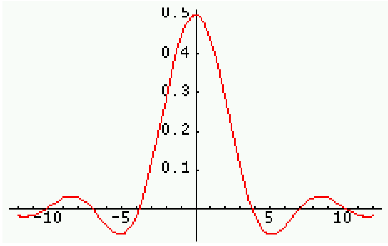
- Peigne de Dirac : $\text{Ш}(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$



- Sinus cardinal : $\text{sinc}(t) = \frac{\sin t}{t}$



- J_1 cardinal : $J_{1c}(t) = \frac{J_1(t)}{t}$



Propriétés générales

Fonction	Transformée	Fonction	Transformée
$a\,f(t) + b\,g(t)$	$a\,\hat{f}(\nu) + b\,\hat{f}(\nu)$	$\frac{df}{dt}$	$2i\pi\nu\hat{f}(\nu)$
$f\left(\frac{t}{a}\right)$	$\lvert a\rvert\,\hat{f}(a\nu)$	$t.f(t)$	$-\frac{1}{2i\pi}\frac{d\hat{f}}{d\nu}$
$\bar{f}(t)$	$\overline{\hat{f}(-\nu)}$	$f(t).g(t)$	$(\hat{f} * \hat{g})(\nu)$
$f(t+\tau)$	$\hat{f}(\nu)\,e^{2i\pi\nu\tau}$ *Animation (340 KB)*	$(f * g)(t)$	$\hat{f}(\nu).\hat{g}(\nu)$
$e^{2i\pi\nu_0 t} f(t)$	$\hat{f}(\nu - \nu_0)$	$h(t) = \int \overline{f(\tau)}\, g(t+\tau) d\tau$	$\hat{h}(\nu) = \overline{\hat{f}(\nu)}\,\hat{g}(\nu)$

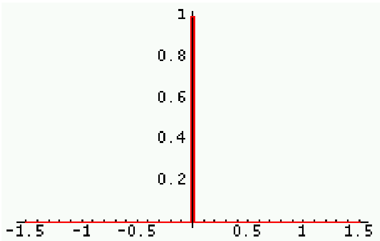
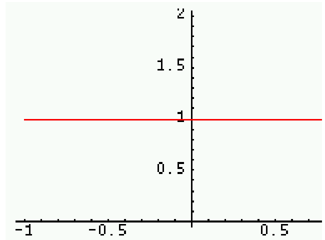
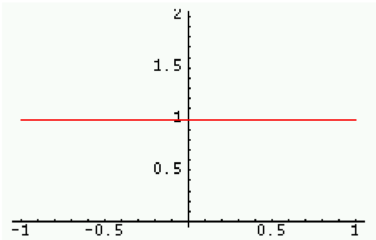
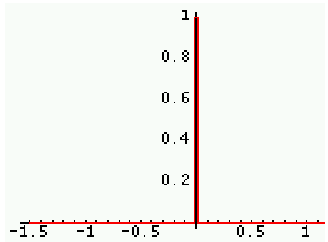
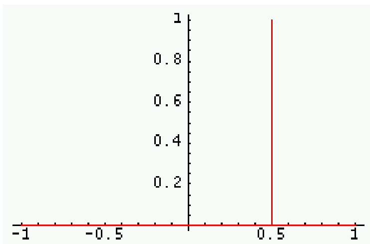
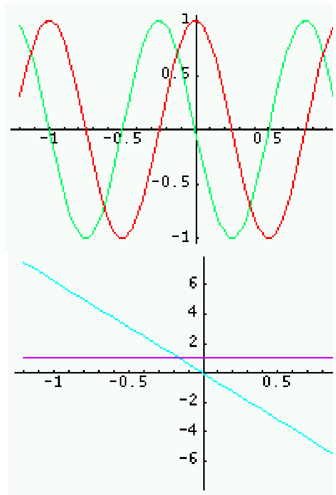
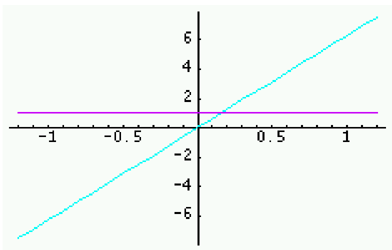
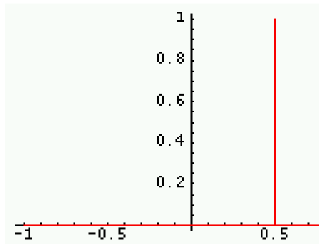
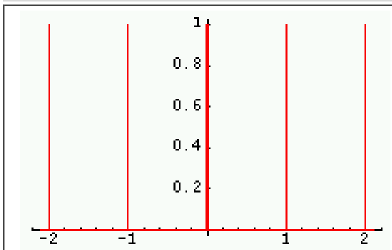
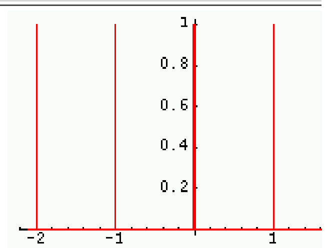
$$\hat{f}(0) = \int_{-\infty}^{\infty} f(t)dt \qquad f(0) = \int_{-\infty}^{\infty} \hat{f}(\nu)d\nu \qquad \hat{\hat{f}}(t) = f(-t)$$

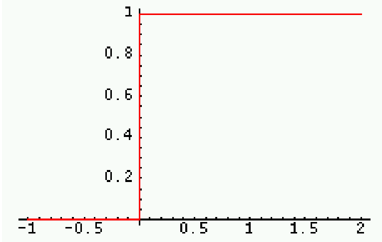
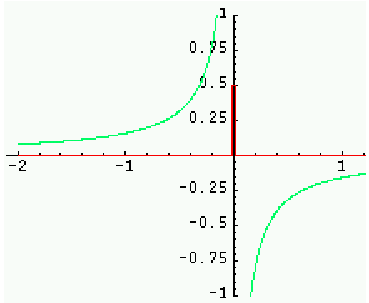
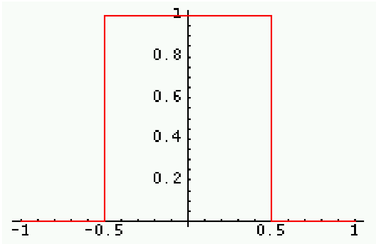
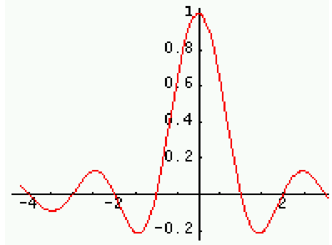
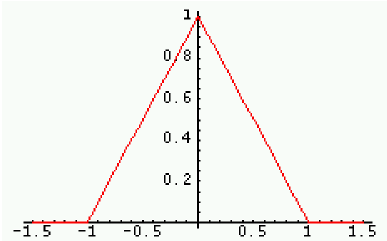
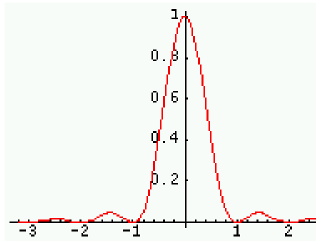
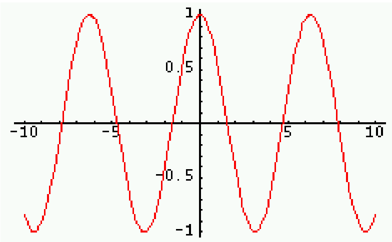
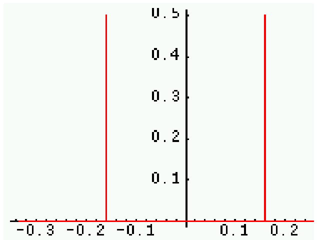
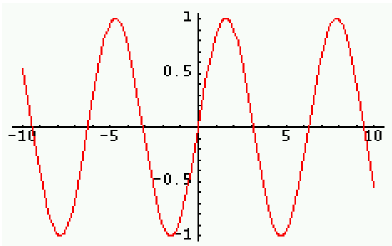
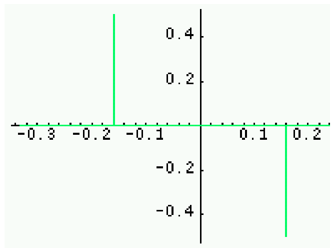
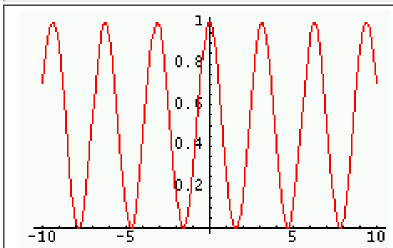
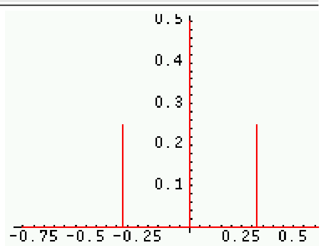
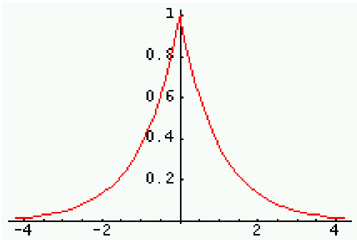
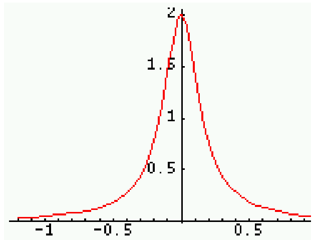
Th. de Parseval $$\int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt = \int_{-\infty}^{\infty} \hat{f}(\nu)\,\overline{\hat{g}(\nu)}\,d\nu$$

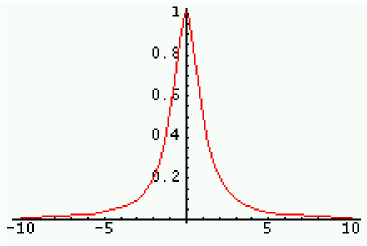
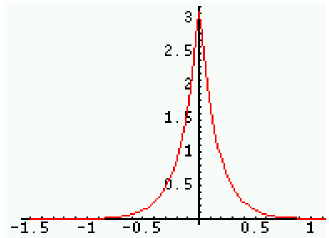
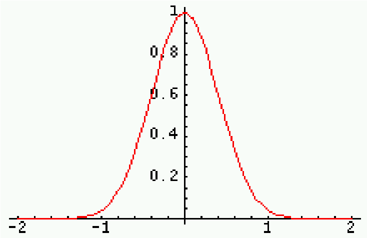
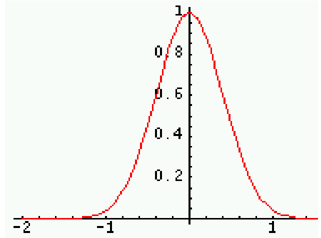
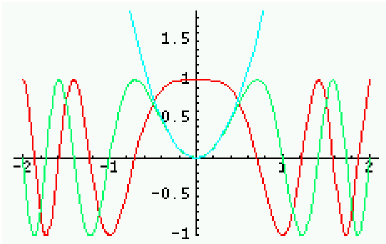
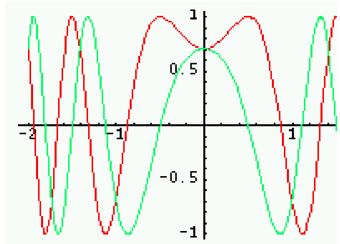
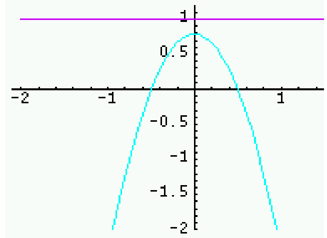
f réelle paire $\Longleftrightarrow$ $\hat{f}$ réelle paire f réelle impaire $\Longleftrightarrow$ $\hat{f}$ imaginaire impaire

f réelle $\iff \hat{f}(-\nu) = \overline{\hat{f}(\nu)}$ (partie réelle paire, partie imaginaire impaire)

Distributions à une dimension

Fonction	Graphe de la fonction *Rouge : partie réelle* ; *Vert : imaginaire* ; *Violet : module* ; *Cyan : phase*	Transformée	Graphe de la TF (parties réelle/imag ; module/
$\delta(t)$		1	
1		$\delta(\nu)$	
$\delta(t-\tau)$		$\exp(-2i\pi\nu\tau)$	
$\exp(2i\pi mt)$		$\delta(\nu-m)$	
$Ш(t)$		$Ш(\nu)$	

$H(t)$		$\frac{1}{2}\delta(\nu) + \mathrm{VP}(\frac{1}{2i\pi\nu})$	
$\Pi(t)$		$\mathrm{sinc}(\pi\nu)$	
$\Lambda(t)$		$\mathrm{sinc}^2(\pi\nu)$	
$\cos(t)$		$\frac{1}{2}\delta(\nu - \frac{1}{2\pi}) + \frac{1}{2}\delta(\nu + \frac{1}{2\pi})$	
$\sin(t)$		$-\frac{i}{2}\delta(\nu - \frac{1}{2\pi}) + \frac{i}{2}\delta(\nu + \frac{1}{2\pi})$	
$\cos^2(t)$		$\frac{1}{2}\delta(\nu) + \frac{1}{4}\delta(\nu - \frac{1}{\pi}) + \frac{1}{4}\delta(\nu + \frac{1}{\pi})$	
$\exp(-\|t\|)$		$\frac{2}{1+4\pi^2\nu^2}$	

$\frac{1}{1+t^2}$		$\pi \exp(-2\pi\lvert\nu\rvert)$	
$\exp(-\pi t^2)$		$\exp(-\pi\nu^2)$	
$\exp(i\pi t^2)$		$e^{i\pi/4} \exp(-i\pi\nu^2)$	 

Distributions à deux dimensions

TF Directe

$$\hat{f}(u,v) = \iint_{-\infty}^{\infty} f(x,y)\ e^{-2i\pi(ux+vy)}\ dxdy$$

TF Inverse

$$f(x,y) = \iint_{-\infty}^{\infty} \hat{f}(u,v)\ e^{2i\pi(ux+vy)}\ dudv$$

Fonctions à variables séparables

$f(x,y) = g(x)\ h(y) \Longleftrightarrow \hat{f}(u,v) = \hat{g}(u)\ \hat{h}(v)$

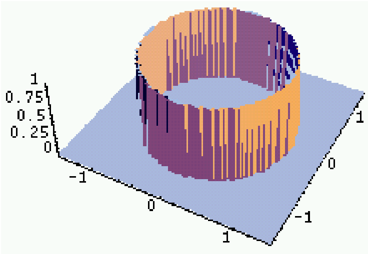
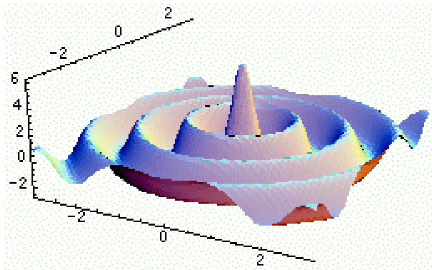
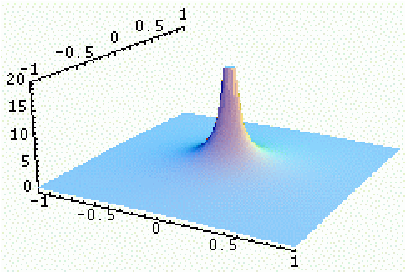
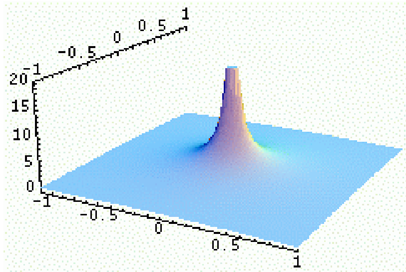
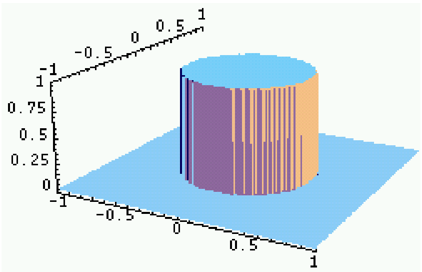
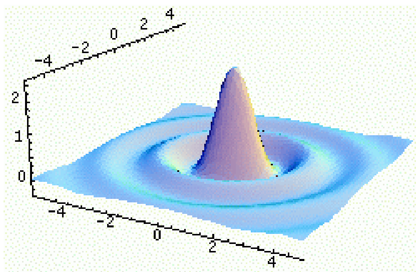
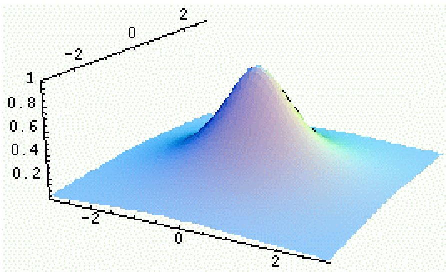
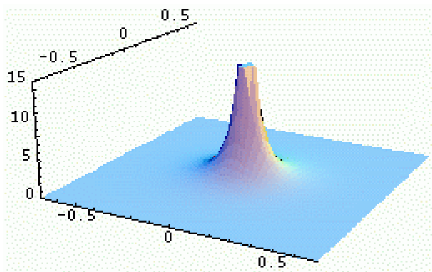
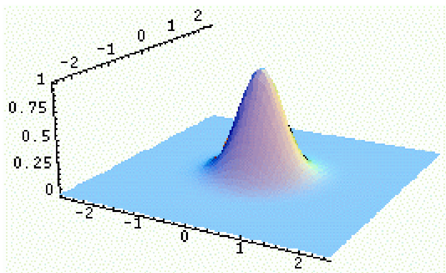
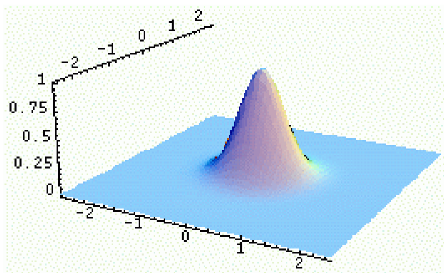
Fonctions à symétrie de révolution

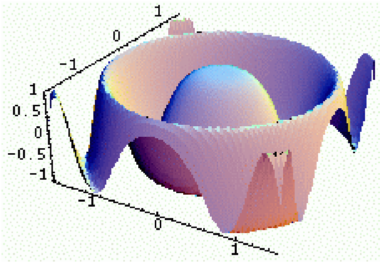
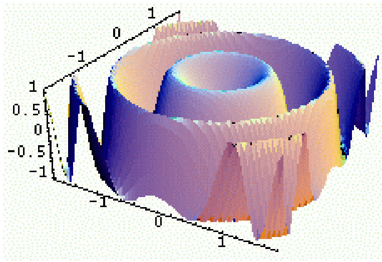
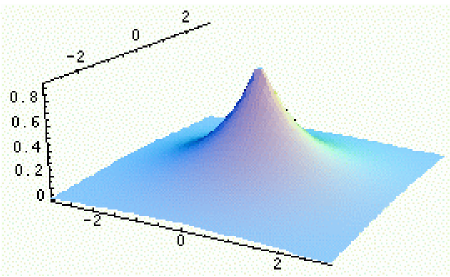
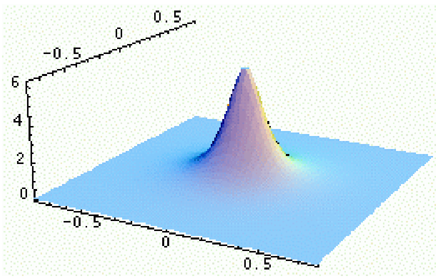
Soit $r = \sqrt{x^2+y^2}$ et $q = \sqrt{u^2+v^2}$. Alors les TF directes et inverses bidimensionnelles sont aussi à symétrie de révolution et s'écrivent à l'aide de la transformée de Hankel

$$\hat{f}(q) = \int_0^\infty f(r)\ J_0(2\pi q r)\ 2\pi r\ dr$$

et

$$f(r) = \int_0^\infty \hat{f}(q)\ J_0(2\pi q r)\ 2\pi q\ dq$$

Fonction	Graphe de la fonction	Transformée	Graphe de la TF
$\delta(r-a)$		$2\pi a J_0(2\pi a q)$	
$1/r$		$1/q$	
$\Pi\left(\frac{r}{d}\right)$		$2\left(\frac{\pi d^2}{4}\right) J_{1c}(\pi d q)$	
$\frac{1}{\sqrt{r^2+a^2}}$		$\frac{\exp(-2\pi a q)}{q}$	
$\exp(-\pi r^2)$		$\exp(-\pi q^2)$	
$\exp\left(i\pi\frac{r^2}{a^2}\right)$	Partie réelle	$ia^2\exp(-i\pi a^2 q^2)$	Partie réelle

			
$\exp(-ar)$		$\dfrac{2\pi a}{(4\pi^2 q^2 + a^2)^{3/2}}$	
$r^2\ f(r)$		$\nabla^2\ \hat{f}$	