

# Large-Scale Structure of Passive Scalar Turbulence

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We investigate the large-scale statistics of a passive scalar transported by a turbulent velocity field by means of direct numerical simulations. We focus on scales larger than the characteristic length scale of scalar injection, yet smaller than the correlation length of the velocity. We show the existence of nontrivial long-range correlations in the form of new power laws for the decay of high-order coarse-grained scalar cumulants. This result contradicts the classical scenario of Gibbs equilibrium statistics that should hold in the absence of scalar flux. The breakdown of “thermal equilibrium” at large scales is traced back to the statistical geometry of turbulent dispersion of two scalar blobs. The numerical values obtained for the scaling exponents of the coarse-grained scalar cumulants are in agreement with recent theoretical results.

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The evolution of a passive scalar field  $\theta(\mathbf{x}, t)$ , like dilute dye concentration or temperature in appropriate conditions, transported by an incompressible velocity field  $\mathbf{v}(\mathbf{x}, t)$ , is governed by the advection-diffusion equation

$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta = \kappa \Delta \theta + f, \quad (1)$$

where  $f$  is a source of scalar fluctuations that acts at a length scale  $L$ . We are interested in the typical situation where the velocity field is turbulent and characterized by a self-similar statistic  $[\mathbf{v}(\mathbf{x} + \mathbf{r}) - \mathbf{v}(\mathbf{x})] \sim r^{1/3}$  in the range of scales delimited above by the velocity correlation length  $L_v$  and below by the viscous scale  $\eta$ . Passive scalar fluctuations are generated at the scale  $L$ , form increasingly finer structures due to velocity advection, and originate a net flux of scalar variance to small scales, where it is eventually smeared out by molecular diffusivity at a scale  $r_d$ . Here we will consider the case where these scales are ordered as follows:  $L_v \gg L \gg \eta, r_d$ . In the range  $L \gg r \gg r_d$  the average scalar flux is constant and equals the average input rate: this is the well studied inertial-convective range where  $\theta$  displays non-Gaussian statistics and anomalous scaling [1,2].

In this Letter we focus our attention on the range of scales larger than  $L$ . There, no scalar flux is present. Accordingly, one would expect Gaussian statistics and equipartition of scalar variance, i.e., the hallmarks of statistical equilibrium. Contrary to this expectation, we show that, in the intermediate range  $L_v \gtrsim r \gtrsim L$ , the “thermal equilibrium” scenario breaks down due to the appearance of new nontrivial power laws for the decay of high-order scalar correlations. These are traced back to the presence of long-range correlations in the dynamics of two scalar blobs advected by the turbulent velocity field. Our results extend the recent findings by Falkovich and Fouxon [3]—obtained in the context of the Kraichnan model of passive scalar advection where the velocity field is Gaussian, self-similar and short-correlated in time—to passive scalar advection by a realistic turbulent flows.

As an instance of a dynamical turbulent flow we consider two-dimensional Navier-Stokes turbulence in the inverse cascade range. This flow has been studied in great detail both experimentally (in fast flowing soap films [4] and in shallow layers of electromagnetically driven electrolyte solutions [5]) and numerically [6,7]. The velocity is statistically homogeneous and isotropic, scale-invariant with exponent  $1/3$  (no intermittency corrections to Kolmogorov scaling) and with dynamical correlation times scaling as  $r^{2/3}$  as expected on dimensional grounds. This flow has also been utilized to investigate passive scalar transport in the range  $L \gtrsim r \gtrsim r_d$  [8] and multiparticle dispersion, an intimately related subject [9].

Let us start the description of our results by recalling that the equilibrium statistics for the scalar field is described by the Gibbs functional  $\mathcal{P}[\hat{\theta}] = Z^{-1} \exp[-\beta \int |\hat{\theta}(\mathbf{k})|^2 d\mathbf{k}]$ . Accordingly, in the range  $kL \lesssim 1$ , where no scalar flux is present, the Fourier modes should behave as independent Gaussian variables with equal variance  $1/(2\beta)$  (equipartition), and the isotropic spectrum  $E(k) = 2\pi k \langle |\hat{\theta}(\mathbf{k})|^2 \rangle$  should be proportional to  $k$ . As shown in Fig. 1, we indeed observe  $E(k) \sim k$  and a statistics of single Fourier modes indistinguishable from Gaussian. However, from those findings alone one cannot state conclusively that large-scale passive scalar is in a thermal equilibrium state, given that they do not allow to rule out the possibility of long-range correlations for higher-order observables (e.g., four-point scalar correlations). A more refined description of the large-scale properties of the passive scalar is thus required. It can be obtained in terms of the coarse-grained field

$$\theta_r(\mathbf{x}, t) = \int G_r(\mathbf{x} - \mathbf{y}) \theta(\mathbf{y}, t) d\mathbf{y} \quad (2)$$

where  $G_r$  acts as a low-pass filter in Fourier space (for instance, the top-hat filter  $G_r(\mathbf{x} - \mathbf{y}) = 1/(\pi r^2)$  if  $|\mathbf{x} - \mathbf{y}| < r$  and zero otherwise; or the Gaussian filter  $G_r(\mathbf{x} - \mathbf{y}) = (2\pi r^2)^{-1} \exp[-|\mathbf{x} - \mathbf{y}|^2/(2r^2)]$ ). For  $r \rightarrow 0$  the filter reduces to a two-dimensional  $\delta$  function and therefore  $\theta_r \rightarrow \theta$ .

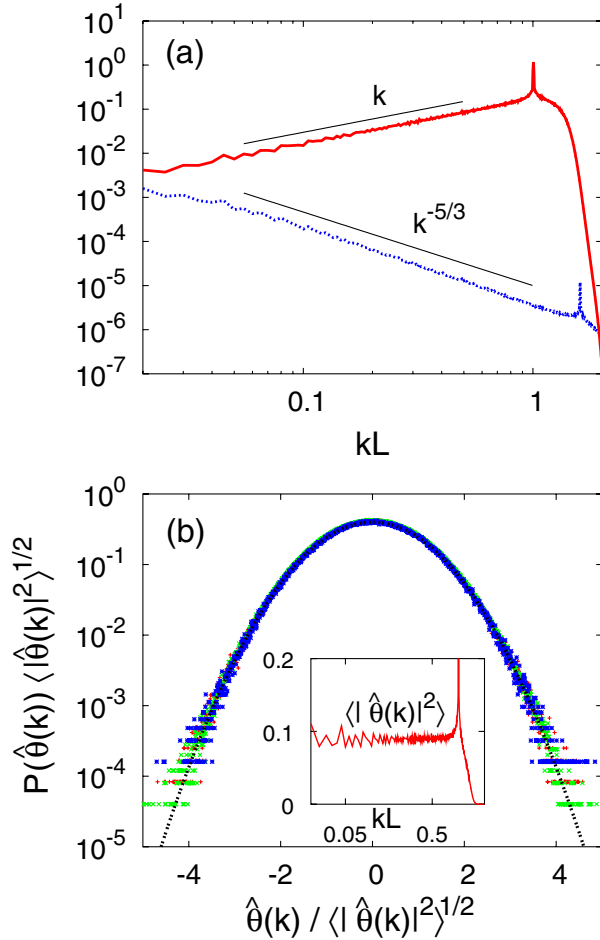


FIG. 1 (color online). (a) Passive scalar (red, upper trace) and velocity (blue, lower trace) spectra. The data result from the time integration of the two-dimensional Navier-Stokes equations  $\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} + \mathbf{F} - \alpha \mathbf{v}$  and Eq. (1) by a pseudospectral method on a  $1024^2$  grid. The term  $-\alpha \mathbf{v}$  is a kinetic energy sink used to model the effect of air friction on thin soap films. The passive scalar is injected by a Gaussian,  $\delta$  correlated in time, statistically homogeneous, and isotropic forcing restricted to a narrow band of wave numbers. The initial condition for the velocity field is a configuration taken from a previous long-time integration and thus already at the statistically stationary state. The passive scalar starts from a zero field configuration, and after a transient of a few large-eddy turnover times  $\tau_v = L_v/v_{rms}$  where  $L_v$  is the integral scale of the velocity field, it reaches its own statistically steady state as well. Time averages are taken after this relaxation time has elapsed, for a total duration of more than  $10^4$  scalar correlation times  $\tau_L \approx \tau_v (L/L_v)^{2/3}$ . Here  $L/L_v \approx 0.02$ . The velocity spectrum agrees with the Kolmogorov prediction  $k^{-5/3}$  and the passive scalar one follows very closely the equipartition spectrum in two-dimensions  $E(k) \sim k$  [see also the inset of panel (b)]. (b) The marginal probability density function of a single Fourier amplitude  $\hat{\theta}(\mathbf{k})$  is indistinguishable from a Gaussian (dotted curve) for all wave numbers in the range  $L^{-1} \gtrsim k \gtrsim L_v^{-1}$ . Here are shown three wave numbers with  $kL = 0.5, 0.25, 0.12$ . In the inset is shown the spectral density  $\langle |\hat{\theta}(\mathbf{k})|^2 \rangle$  that shows a neat plateau at  $kL \ll 1$  (notice the linear scale on the vertical axis).

The statistics of  $\theta$  is typically super-Gaussian [2]: its probability density function has exponential-like tails even for a Gaussian,  $\delta$  correlated in time driving force  $f$ . Indeed, in the latter case it can be shown exactly, by a minor modification of the arguments given in Ref. [10], that  $\theta$  is the product of two independent random variables  $\theta \triangleq \phi \sqrt{F_0 T}$  where  $\phi$  is a Gaussian variable of zero mean and unit variance,  $F_0$  is the average injection rate of scalar fluctuations, and  $T$  is a positive-defined random variable, independent from  $\phi$ . The variable  $T$  is essentially the time taken by a spherical blob of minute initial size to disperse across a length  $L$  for a given flow configuration [11]. Therefore  $\langle \theta^{2n} \rangle / \langle \theta^2 \rangle^n = (2n-1)!! \langle T^n \rangle / \langle T \rangle^n \geq (2n-1)!!$ ; i.e.,  $\theta$  is super-Gaussian unless  $T$  is nonrandom. The distribution of  $\theta_r$  is super-Gaussian as well; however, as  $r$  increases above the forcing correlation length, the probability density of  $\theta_r$  tends to a Gaussian distribution, as it is clearly seen by the scale dependence of the distribution flatness and hyperflatness (see Fig. 2).

Within the framework of Gibbs statistical equilibrium, the scalar field has vanishingly small correlations above the scale  $L$ : therefore one could view  $\theta_r$  as the sum of  $N \approx (r/L)^2$  independent random variables (identically distributed as  $\theta$ ) divided by  $N$ . By central-limit-theorem arguments [12], the moments of order  $2n$  of the coarse-grained scalar field (odd-order moments vanish by symmetry) should then scale as  $N^{-n}$ , giving  $\langle (\theta_r)^{2n} \rangle \sim \langle \theta^2 \rangle^n \times (r/L)^{-2n}$ . This is a very good estimate for  $n=1$ : indeed, as shown in Fig. 3, the product  $(r/L)^2 \langle \theta_r^2 \rangle$  has a very neat plateau. This is consistent with the fast decay of the two-point scalar correlation  $\langle \theta(\mathbf{x}, t) \theta(\mathbf{x} + \mathbf{r}, t) \rangle$  at  $r \gtrsim L$ . Indeed, in this case the second-order moment  $\langle \theta_r^2 \rangle = \int d\mathbf{y}_1 d\mathbf{y}_2 G_r(\mathbf{y}_1 - \mathbf{x}) G_r(\mathbf{y}_2 - \mathbf{x}) \langle \theta(\mathbf{y}_1, t) \theta(\mathbf{y}_2, t) \rangle$  is dominated by contributions with  $|\mathbf{y}_1 - \mathbf{y}_2| \lesssim L$  yielding  $\langle \theta_r^2 \rangle \sim \langle \theta^2 \rangle (r/L)^{-2}$ . Alternatively, by Fourier transforming the

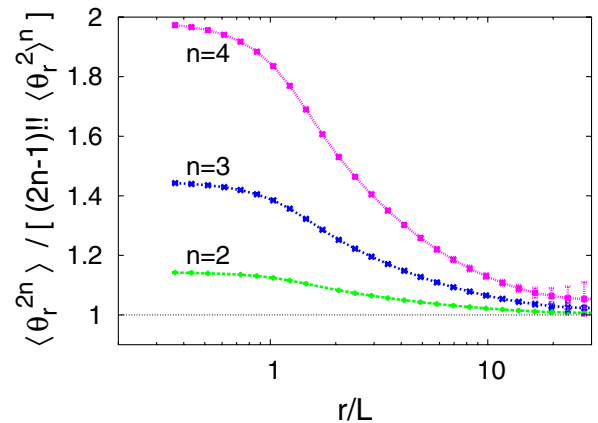


FIG. 2 (color online). The flatness and hyperflatness of the coarse-grained scalar field as a function of  $r/L$ , normalized by their Gaussian values  $(2n-1)!!$ . For  $r \rightarrow 0$  the curves tend to the flatness factors of the field  $\theta$ : the numerical values correspond to a super-Gaussian probability density function  $\ln p(\theta) \sim -\theta^{1.6}$ .

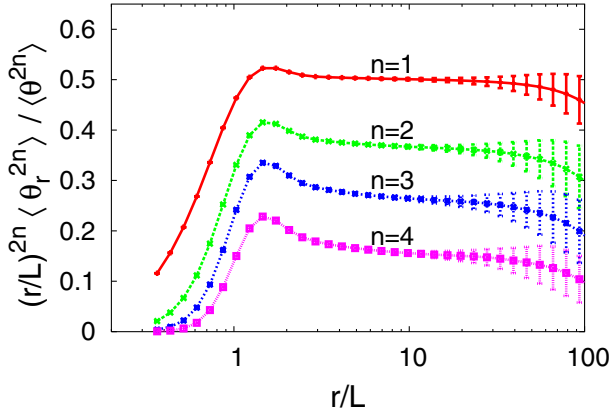


FIG. 3 (color online). Moments of the coarse-grained scalar field  $\langle \theta_r^{2n} \rangle$  compensated by the thermal equilibrium expectation  $(r/L)^{2n}$ . The error bars are determined by dividing the sample in ten subsamples and computing the dispersion around the mean.

coarse-grained field one obtains  $\langle \theta_r^2 \rangle = \int |\hat{G}_r(\mathbf{k})|^2 \times \langle |\hat{\theta}(\mathbf{k}, t)|^2 \rangle d\mathbf{k} \simeq \langle \theta^2 \rangle (r/L)^{-2}$ , since the transformed filter  $\hat{G}_r(\mathbf{k})$  is close to unity for  $kr \ll 1$  and falls off very rapidly for  $kr \gg 1$ , and  $|\hat{\theta}(\mathbf{k}, t)|^2 \simeq \langle \theta^2 \rangle / (\pi L^2)$ . In summary, two-point statistics appears to be consistent with Gibbs equilibrium ensemble. The situation for multipoint correlations will turn out to be different.

A careful inspection of higher-order moments shows a less good agreement with central-limit-theorem estimates (see Fig. 3): this points to the existence of subleading contributions to the moments  $\langle \theta_r^{2n} \rangle$  for  $n > 1$  arising from long-range correlations of multiple scalar products. In order to quantify more precisely the rate of convergence to Gaussianity and its relationship to long-range correlations, it is useful to consider the cumulants of the random variable  $\theta_r$ . According to the central-limit theorem [12], the cumulant of order  $2n$  should vanish with  $N^{-2n+1}$  leading to an expected scaling  $\langle \langle \theta_r^{2n} \rangle \rangle \sim \langle \langle \theta_r^2 \rangle \rangle (r/L)^{-4n+2}$ . Let us reiterate that the former expression is expected to be valid in absence of scalar correlations across length scales  $r \gtrsim L$ .

For  $n = 1$  we have  $\langle \langle \theta_r^2 \rangle \rangle = \langle \theta_r^2 \rangle$  whose behavior has been already detailed above. In Fig. 4 we show the behavior of  $\langle \langle \theta_r^4 \rangle \rangle = \langle \theta_r^4 \rangle - 3\langle \theta_r^2 \rangle^2$  and  $\langle \langle \theta_r^6 \rangle \rangle = \langle \theta_r^6 \rangle - 15\langle \theta_r^2 \rangle \times \langle \theta_r^4 \rangle + 30\langle \theta_r^2 \rangle^3$ . For the fourth-order cumulant, we observe a scaling law very close to the theoretical expectation  $\langle \langle \theta_r^4 \rangle \rangle \simeq \langle \langle \theta_r^4 \rangle \rangle (r/L)^{-16/3}$  obtained in Ref. [3] for  $\gamma = 2/3$ , which corresponds to Kolmogorov-Richardson scaling for the velocity dynamics. This has to be contrasted with the scaling law  $(r/L)^{-6}$  given by central-limit arguments. The breakdown of the central-limit theorem is due to the existence of long-range dynamical correlations in the range  $r \gg L$ . These exclude the possibility of a true Gibbs statistical equilibrium at large scales. The leading contribution to the fourth-order cumulant  $\langle \langle \theta_r^4 \rangle \rangle = \int dy_1 \times dy_2 dy_3 dy_4 G_r(y_1 - \mathbf{x}) G_r(y_2 - \mathbf{x}) G_r(y_3 - \mathbf{x}) G_r(y_4 - \mathbf{x}) \times \langle \langle \theta(y_1, t) \theta(y_2, t) \theta(y_3, t) \theta(y_4, t) \rangle \rangle$  comes from configurations with the four points arranged in two pairs of close

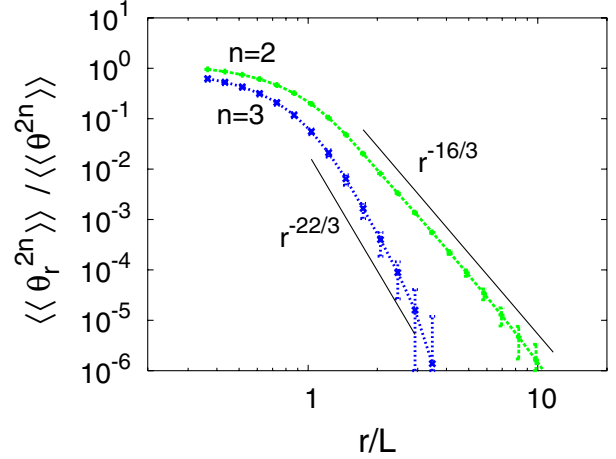


FIG. 4 (color online). Cumulants of order 4 and 6 for the coarse-grained scalar field. The best fits for the slopes give exponents  $5.32 \pm 0.05$  for  $n = 2$  and  $7.5 \pm 0.5$  for  $n = 3$ . The theoretical values  $-16/3$  and  $-22/3$  are shown for comparison.

particles (e.g.,  $|y_1 - y_2| \lesssim L$  and  $|y_3 - y_4| \lesssim L$ ) separated by a distance  $r$  (e.g.,  $|y_1 - y_3| \simeq r$ ). Otherwise stated, two-point correlators of the squared scalar field  $\langle \langle \theta^2(\mathbf{x}, t) \theta^2(\mathbf{x} + \mathbf{r}, t) \rangle \rangle$  must display a nontrivial scaling  $(r/L)^{-4/3}$ . We will get back to the issue of the statistics of  $\theta^2$  momentarily. The sixth-order cumulant  $\langle \langle \theta_r^6 \rangle \rangle$  is extremely difficult to measure because of the strong cancellations between various terms. Upon collecting the statistics over about ten thousand scalar correlation times, we can conclude that the results are consistent with the power-law decay  $\langle \langle \theta_r^6 \rangle \rangle \simeq \langle \langle \theta_r^6 \rangle \rangle (r/L)^{-22/3}$  suggested by the theory for  $\gamma = 2/3$ , and arising from terms like  $\langle \langle \theta_r^4 \rangle \rangle \times \langle \langle \theta_r^2 \rangle \rangle$  that appear in the expansion of the sixth-order cumulant [3]. The actual exponent for  $\langle \langle \theta_r^6 \rangle \rangle$  cannot be de-

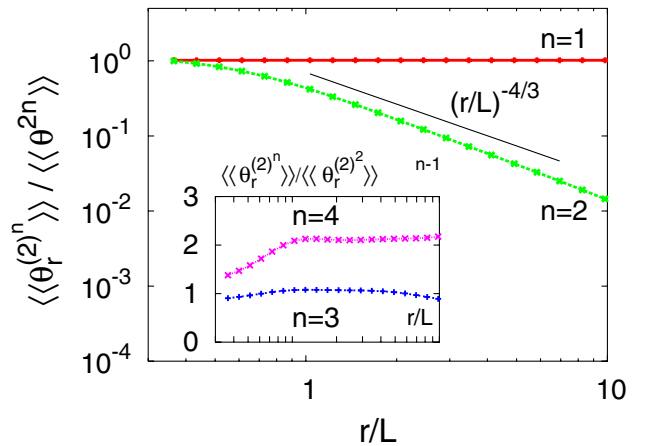


FIG. 5 (color online). Cumulants of the coarse-grained, squared scalar field  $\theta_r^{(2)}$ . The definitions for the low-order cumulants of a generic random variable  $z$  are:  $\langle \langle z \rangle \rangle = \langle z \rangle$ ,  $\langle \langle z^2 \rangle \rangle = \langle z^2 \rangle - \langle z \rangle^2$ ,  $\langle \langle z^3 \rangle \rangle = \langle z^3 \rangle - 3\langle z \rangle \langle z^2 \rangle + 2\langle z \rangle^3$ ,  $\langle \langle z^4 \rangle \rangle = \langle z^4 \rangle - 4\langle z \rangle \langle z^3 \rangle - 3\langle z^2 \rangle^2 + 12\langle z^2 \rangle \langle z \rangle^2 - 6\langle z \rangle^4$ .

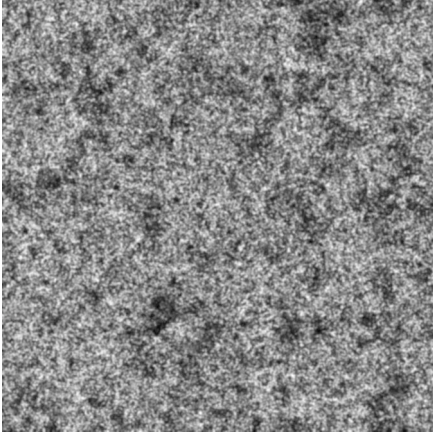


FIG. 6. A snapshot of the squared scalar field  $\theta^2$ . Remark the inhomogeneous distribution of scalar intensity originating from long-range correlations  $\langle\langle\theta_r^{(2)^2}\rangle\rangle \sim r^{-4/3}$ .

terminated with great precision, yet it lies within the range between  $-7$  and  $-8$ , thus definitely different from the central-limit-theorem expectation,  $-10$ .

Further insight on the deviations from statistical equilibrium at large scales can be gained by studying the statistics of the coarse-grained squared scalar field

$$\theta_r^{(2)}(\mathbf{x}, t) = \int G_r(\mathbf{x} - \mathbf{y}) \theta^2(\mathbf{y}, t) d\mathbf{y}. \quad (3)$$

The cumulants of  $\theta_r^{(2)}$  give useful information about the presence of long-range correlations of the field  $\theta^2$ . The first-order cumulant  $\langle\langle\theta_r^{(2)}\rangle\rangle \equiv \langle\theta_r^{(2)}\rangle$  is trivially equal to  $\langle\theta^2\rangle$ . The second-order cumulant  $\langle\langle\theta_r^{(2)^2}\rangle\rangle = \langle\theta_r^{(2)^2}\rangle - \langle\theta_r^{(2)}\rangle^2$  for a scalar field in thermal equilibrium should decay rapidly to zero at large scales  $r \gtrsim L$ . On the contrary, as shown in Fig. 5, we observe a slow power-law decay with an exponent close to the theoretical expectation (for  $\gamma = 2/3$ )  $\langle\langle\theta_r^{(2)^2}\rangle\rangle \simeq \langle\langle\theta^4\rangle\rangle (r/L)^{-4/3}$  [3].

Higher-order cumulants behave self-similarly as  $\langle\langle\theta_r^{(2)^n}\rangle\rangle \sim \langle\langle\theta_r^{(2)^2}\rangle\rangle^{n-1}$ . This result can be interpreted in terms of the geometrical properties of the positive measure defined by the squared scalar field: at scales  $r \gtrsim L$  the field  $\theta^2$  appears as a purely fractal object with dimension  $D_F \approx 2/3$  (see Fig. 6) onto a space-filling background.

We end up by discussing the physical origin of long-range scalar correlations. For a Gaussian forcing we have  $\theta^2(\mathbf{x}_1, t) \theta^2(\mathbf{x}_2, t) \stackrel{\Delta}{=} F_0^2 \phi_1^2 \phi_2^2 T_1 T_2$ . At distances  $|\mathbf{x}_1 - \mathbf{x}_2| = r \gg L$  the two Gaussian variables  $\phi_1$  and  $\phi_2$  are independent. However, this is not the case for  $T_1$  and  $T_2$  because of the underlying velocity field. Therefore, the long power-law tail for  $\langle\langle\theta_r^{(2)^2}\rangle\rangle$  arises from events where  $\langle T_1 T_2 \rangle \gg \langle T \rangle^2$ . This amounts to say that two blobs of initial size

smaller than  $L$ , released at a distance  $r \gg L$  in the same flow, do not spread considerably by turbulent diffusion (i.e.,  $T_{1,2} \gg \langle T \rangle$ ) with a probability  $\sim (r/L)^{-4/3}$ .

In summary, we have shown that the scenario of Gibbs statistical equilibrium is not valid for large-scale passive scalar turbulence in spite of the absence of scalar flux. Long-range correlations appear at the level of high-order cumulants of the coarse-grained scalar field. It would be extremely interesting to understand whether the breakdown of “thermal equilibrium” holds for other turbulent systems as well, in particular, for two and three-dimensional hydrodynamic turbulence at very large scales.

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  - [12] The statistics of the random variable  $z_N = (\sum_i^N w_i)/N$ , where  $w_i$  are independent and identically distributed random variables with zero mean and finite variance  $\sigma^2$ , is completely characterized by the generating function  $G(s) = \langle \exp(s z_N) \rangle$ . By virtue of statistical independency and identity in distribution of the  $w$ 's, we have  $G(s) = \langle \prod_i^N \exp(s w_i / N) \rangle = \langle \exp(s w / N) \rangle^N = g(s/N)^N$  where  $g$  is the generating function for  $w$ . For  $N \gg 1$  we have  $g(s/N) \simeq (1 + s^2 \sigma^2 / (2N^2))^N \simeq \exp(s^2 \sigma^2 / (2N))$  (a version of the central-limit theorem) and therefore  $\langle z_N^{2n} \rangle \simeq (2n-1)!! \sigma^{2n} N^{-n}$ . The cumulants  $\langle\langle z_N^{2n} \rangle\rangle$  are defined in terms of the Taylor series of  $\ln G(s)$  around  $s = 0$ : since  $\ln G(s) = N \ln g(s/N)$  we have  $\langle\langle z_N^{2n} \rangle\rangle \sim \langle\langle w^{2n} \rangle\rangle N^{-2n+1}$ .