Ex 2

Déterminant et trace

1) une matrice $A(t)$; $\frac{dA(t)}{dt} = A(t)\,B$ (B, matrice)

Montrer que $A(t) = A(0)\,e^{Bt}$

Definition : $e^{Bt} = \sum_{n=0}^{\infty} \frac{(Bt)^n}{n!}$

$$\Rightarrow \frac{d}{dt} e^{Bt} = \sum_{n=0}^{\infty} n \frac{B^n t^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{B^n t^{n-1}}{(n-1)!} =$$

$$= B \sum_{n=1}^{\infty} \frac{(Bt)^{n-1}}{(n-1)!} = B \sum_{p=0}^{\infty} \frac{(Bt)^p}{p!} = B e^{Bt} \quad \left(= e^{Bt} B \right)$$

$(p = n-1)$

$$\left.\begin{aligned} &1 : \frac{dA(t)}{dt} = A(0)\,B\,e^{Bt} = A(0)\,e^{Bt}\,B \\ &2 : A(t)\,B = A(0)\,e^{Bt}\,B \end{aligned}\right\} \text{cqfd}$$

La solution de : $\frac{dA(t)}{dt} = B\,A(t)$?

test logique : $A(t) = e^{Bt} A(0)$

$\Rightarrow \frac{dA(t)}{dt} = B\,e^{Bt} A(0) = B\,A(t)$, ça marche !

2) Montrer que : $\det e^{At_1} \times \det e^{At_2} = \det e^{A(t_1+t_2)}$

Considerons $e^A \times e^B$; $e^B \times e^A$; $e^{(A+B)}$

$$\begin{cases} e^A e^B = \sum_n \frac{A^n}{n!} \sum \frac{B^m}{m!} = \sum_{n,m} \frac{A^n B^m}{n!m!} = (1) \\ e^B e^A = \sum_{n,m} \frac{B^n A^m}{n!m!} = (2) \\ e^{(A+B)} = \sum_n \frac{(A+B)^n}{n!} = (3) \end{cases}$$

En general, on ne peut pas supposer : $(1) = (2) = (3)$

Mais, si A et B commute : $[A, B] = 0$

$\Rightarrow e^A e^B = e^B e^A = e^{(A+B)}$

$[At_1, At_2] = At_1 At_2 - At_2 At_1 = A^2 t_1 t_2 - A^2 t_1 t_2 = 0$

$\Rightarrow e^{At_1} \times e^{At_2} = e^{A(t_1+t_2)}$

$\Rightarrow \det\left(e^{At_1} \times e^{At_2}\right) = \det\left(e^{A(t_1+t_2)}\right)$

utilisons : $\det AB = \det A \times \det B$

$\Rightarrow \det\left(e^{At_1}\right) \times \det\left(e^{At_2}\right) = \det\left(e^{A(t_1+t_2)}\right)$

cqfd

Deduire : $\det e^A = e^{\operatorname{Tr} A}$

suggestion : $g(t) \equiv \det\left(e^{At}\right)$ et prend la differentielle

Définition d'une différentielle :

$$\frac{d g(t)}{dt} = \frac{1}{\delta t}\Big(g(t+\delta t) - g(t)\Big) =$$

$$= \frac{1}{\delta t}\left(\det e^{A(t+\delta t)} - \det e^{At}\right) =$$

$$= \frac{1}{\delta t}\left(\det e^{A\delta t} - 1\right)\det e^{At} =$$

$$= \frac{1}{\delta t}\left(\det e^{A\delta t} - 1\right) g(t)$$

$$\det e^{A\delta t} = \det\left(\mathbb{1} + A\delta t + \ldots (\delta t)^n\right) \approx \det\left(\mathbb{1} + A\delta t\right)$$

exemple : 2x2 matrice :

$$\det e^{A\delta t} = \det\left(\mathbb{1} + A\delta t\right) = \begin{vmatrix} 1 + A_{11}\delta t & A_{12}\delta t \\ A_{21}\delta t & 1 + A_{22}\delta t \end{vmatrix} =$$

$$= 1 + \delta t\left(A_{11} + A_{22}\right) + \underbrace{(\delta t)^2}_{\approx 0}\left(A_{11}A_{22} - A_{21}A_{12}\right) \approx$$

$$\approx 1 + \delta t \times \mathrm{Tr}\, A$$

$$\Rightarrow \frac{d g(t)}{dt} = \frac{1}{\delta t}\left(\cancel{1} + \delta t \times \mathrm{Tr}\, A - \cancel{1}\right) g(t) = \mathrm{Tr}\, A \times g(t)$$

une équation différentielle !

$$\Rightarrow g(t) = K e^{\mathrm{Tr} A \times t} + C$$

$$t = 0 \Rightarrow g(t) = 1 \Rightarrow K = 1 \text{ et } C = 0$$

$$\Rightarrow g(t) = e^{\widetilde{\mathrm{Tr} A \times t}} = \det\left(e^{\widetilde{A \times t}}\right) \quad \underline{\text{cqfd}}$$