Box 2.2: The spectral decomposition – important!

The *spectral decomposition* is an extremely useful representation theorem for normal operators.

Theorem 2.1: **(Spectral decomposition)** Any normal operator M on a vector space V is diagonal with respect to some orthonormal basis for V. Conversely, any diagonalizable operator is normal.

Proof

The converse is a simple exercise, so we prove merely the forward implication, by induction on the dimension d of V. The case $d = 1$ is trivial. Let λ be an eigenvalue of M, P the projector onto the λ eigenspace, and Q the projector onto the orthogonal complement. Then $M = (P + Q)M(P + Q) = PMP + QMP + PMQ + QMQ$. Obviously $PMP = \lambda P$. Furthermore, $QMP = 0$, as M takes the subspace P into itself. We claim that $PMQ = 0$ also. To see this, let $|v\rangle$ be an element of the subspace P. Then $MM^\dagger|v\rangle = M^\dagger M|v\rangle = \lambda M^\dagger|v\rangle$. Thus, $M^\dagger|v\rangle$ has eigenvalue λ and therefore is an element of the subspace P. It follows that $QM^\dagger P = 0$. Taking the adjoint of this equation gives $PMQ = 0$. Thus $M = PMP + QMQ$. Next, we prove that QMQ is normal. To see this, note that $QM = QM(P + Q) = QMQ$, and $QM^\dagger = QM^\dagger(P + Q) = QM^\dagger Q$. Therefore, by the normality of M, and the observation that $Q^2 = Q$,

$$QMQ\,QM^\dagger Q = QMQM^\dagger Q \tag{2.37}$$

$$= QMM^\dagger Q \tag{2.38}$$

$$= QM^\dagger MQ \tag{2.39}$$

$$= QM^\dagger QMQ \tag{2.40}$$

$$= QM^\dagger Q\,QMQ\,, \tag{2.41}$$

so QMQ is normal. By induction, QMQ is diagonal with respect to some orthonormal basis for the subspace Q, and PMP is already diagonal with respect to some orthonormal basis for P. It follows that $M = PMP + QMQ$ is diagonal with respect to some orthonormal basis for the total vector space. □

In terms of the outer product representation, this means that M can be written as $M = \sum_i \lambda_i |i\rangle\langle i|$, where λ_i are the eigenvalues of M, $|i\rangle$ is an orthonormal basis for V, and each $|i\rangle$ an eigenvector of M with eigenvalue λ_i. In terms of projectors, $M = \sum_i \lambda_i P_i$, where λ_i are again the eigenvalues of M, and P_i is the projector onto the λ_i eigenspace of M. These projectors satisfy the completeness relation $\sum_i P_i = I$, and the orthonormality relation $P_i P_j = \delta_{ij} P_i$.