# 1 Hydrostatic pressure

Fig. 1 shows a manometer, which is a U-shaped tube containing mercury at density $\rho_m$. Manometers are used as pressure measuring devices. If the fluid in the tank A has a pressure $p$ and density $\rho$, then show that the gauge pressure in the tank is

$$p - p_{atm} = \rho_m g h - \rho g a$$

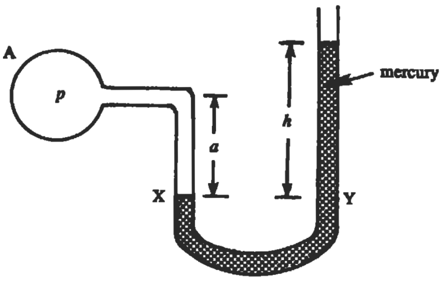


Figure 1: A mercury manometer.

# 2 Surface tension and Laplace law

You are blowing a soap bubble with a straw. The bubble is 5 cm in diameter. What is the overpressure inside the bubble ? You are now connecting two bubbles with a straw so that air can go from one bubble to the other one. The two bubbles are 1 cm and 5 cm in diameter respectively. What happens to the system ? Take $\gamma = 50$mN.m$^{-1}$ for surface tension.

# 3 Surface tension and energy

A shaving foam is made of bubbles 10 microns in diameter dispersed inside a liquid. Calculate the number of bubbles inside 1 cm$^3$ of foam assuming that the amount of liquid can be neglected. What is the total surface energy assuming that the bubbles are spherical ? Take $\gamma = 50$mN.m$^{-1}$ for surface tension.

# 4 Capillary rise

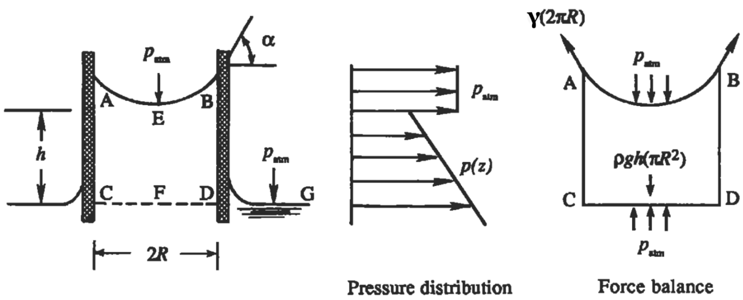


Figure 2: Rise of the liquid in a narrow tube.

Estimate the height to which water at 20° will rise in a capillary glass tube 3 mm in diameter exposed to the atmosphere. For the water in contact with glass the wetting angle $\alpha$ is nearly 90°C. At 20°C and water-air combination, $\gamma = 0.073$ N.m$^{-1}$.

## 5 Viscous stress

Consider the viscous flow in a channel of width $2b$. The channel is aligned in the $x$ direction, and the velocity at a distance $y$ from the centerline is given by the parabolic distribution

$$u(y) = U_0 \left[1 - \frac{y^2}{b^2}\right]$$

In terms of the viscosity $\eta$, calculate the shear stress at a distance of $y = b/2$.

## 6 Orifice in a tank

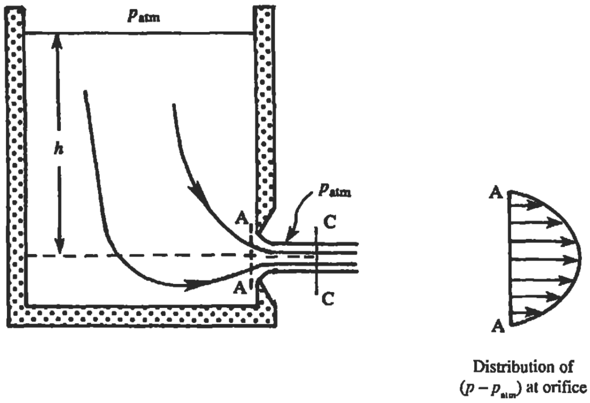


Figure 3: Flow through a sharp-edged orifice. Pressure has the atmospheric value everywhere across section CC; its distribution across orifice AA is indicated.

A classical application of Bernoulli's equation is the flow through an orifice or opening in a tank (fig .3).
What are the three conditions to consider that $\frac{1}{2}\|\vec{u}\|^2 + p/\rho + gz$ is constant throughout the flow ?
Why is it better to apply Bernouilli's equation between the surface and a point at C ?
Give the jet velocity in C and the mass flow rate of the water leaving the tank. Note $A_C$ the area of the jet in C.

[Remark: For orifices having a sharp edge, $A_C$ has been found to be $\approx 62\%$ of the orifice area. If the orifice has a well-rounded opening, the jet does not contract, the streamlines right at the exit are parallel, the pressure at the exit is uniform and equal the atmospheric pressure.]

## References

[1] Fluid Mechanics, P. K. Kundu and I. M. Cohen, 4th edition, Academic Press
[2] The National Committee for Fluid Mechanics Films movie series
https://www.youtube.com/playlist?list=PL96364257FFBC4A3A
[3] Hydrodynamique physique, E. Guyon, J.-P. Hulin and L. Petit, EDP SCIENCES

# 1 Cartesian tensors

## 1.1 Force on a surface

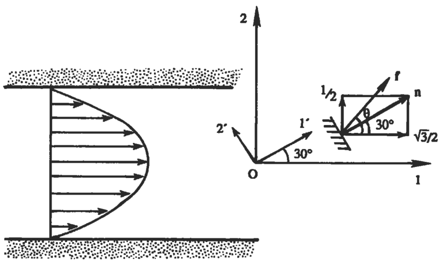

Figure 4: Determination of force on an area element.

Consider a two-dimensional parallel flow through a channel. Take $x_1$, $x_2$ as the coordinate system, with $x_1$ parallel to the flow (Fig. 4). The viscous stress tensor at a point in the flow has the form

$$\overline{\overline{\sigma}} = \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix}$$

where the constant $\tau$ is positive in one half of the channel, and negative in the other half. Find the magnitude and direction of force $\vec{f}$ per unit area on an element whose outward normal points at 30° to the direction of the flow.

## 1.2 Strain rate tensor

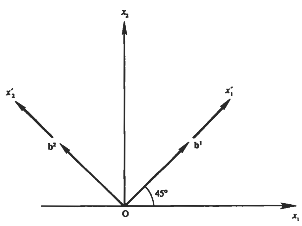

Figure 5: Original coordinate system $\mathrm{O}x_1x_2$ and rotated coordinate system $\mathrm{O}x_1'x_2'$ coinciding with the eigenvectors.

The strain rate tensor $\overline{\overline{e}}$ is related to the velocity vector $\vec{u}$ by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

For a two-dimensional parallel flow

$$\vec{u} = \begin{bmatrix} u_1(x_2) \\ 0 \end{bmatrix},$$

Show how $\overline{\overline{e}}$ is diagonalized in the frame of reference coinciding with the principal axes.
What is the condition for a fluid element to rotate inside the flow ?

### 1.3 Formal analysis - 1

Using indicial notation, show that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

[Hint: call $\vec{d} = \vec{b} \times \vec{c}$. Then $(\vec{a} \times \vec{d})_m = \varepsilon_{pqm} a_p d_q = \varepsilon_{pqm} a_p \varepsilon_{ijq} b_i c_j$. Using $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, show that $(\vec{a} \times \vec{d})_m = (\vec{a}.\vec{c})b_m - (\vec{a}.\vec{b})c_m$ .]

### 1.4 Formal analysis - 2

Show that the condition for the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ to be coplanar is

$$\varepsilon_{ijk} a_i b_j c_k = 0$$

### 1.5 Formal analysis - 3

Prove the following relationships:

$$\delta_{ij}\delta_{ij} = 3$$

$$\varepsilon_{pqr}\varepsilon_{pqr} = 6$$

$$\varepsilon_{pqi}\varepsilon_{pqj} = 2\delta_{ij}$$

### 1.6 Formal analysis - 4

Prove that $\text{div}(\vec{\text{curl}}(\vec{u})) = 0$ for any vector $\vec{u}$ regardless of the coordinate system.
[Hint: use the vector integral theorems]

### 1.7 Formal analysis - 5

Prove that $\vec{\text{curl}}(\vec{\text{grad}}(\phi)) = \vec{0}$ for any single-valued scalar $\phi$ regardless of the coordinate system.
[Hint: use Stokes theorems]

## 2 Kinematics

Differences between streamlines, streaklines and pathlines
https://en.wikipedia.org/wiki/Streamlines,_streaklines,_and_pathlines

### 2.1 Exercise

A two-dimensional steady flow has velocity components

$$u = y \qquad v = x$$

Show that the streamlines are rectangular hyperbolas

$$x^2 - y^2 = \text{constant}$$

Sketch the flow pattern, and convince yourself that it represents an irrotational flow in a 90° corner.
Is the flow compressible or incompressible ?

### 2.2 Exercise

Consider a steady axisymmetric flow of a compressible fluid. The equation of continuity in cylindrical coordinates $(r, \theta, z)$ is

$$\frac{\partial}{\partial r}(\rho r u_r) + \frac{\partial}{\partial z}(\rho r u_z) = 0$$

Show how we can define a streamfunction so that the equation of continuity is satisfied automatically.

### 2.3 Exercise

If a velocity field is given by $u = ay$, compute the circulation around a circle of radius $r = 1$ about the origin. Check the result by using Stokes' theorem.

## 2.4 Exercise

1) Consider a plane Couette flow of a viscous fluid confined between two flat plates at a distance $b$ apart (Fig. 6). At steady state the velocity distribution is

$$u = Uy/b \qquad v = w = 0,$$

where the upper plate at $y = b$ is moving parallel to itself at speed $U$, and the lower plate is held stationary. Find the rate of linear strain, the rate of shear strain, and vorticity. Show that the streamfunction is given by

$$\psi = \frac{Uy^2}{2b} + \text{constant}$$

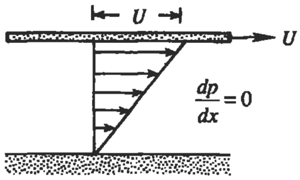


Figure 6: Plane Couette flow.

2) Show that the vorticity for a plane flow on the $xy$-plane is given by $\omega_z = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)$.
Using this expression, find the vorticity for the flow in (1).

## 2.5 Streamlines versus pathlines

The velocity components in an unsteady flow are given by

$$u = \frac{x}{1+t} \text{ and } v = \frac{2y}{2+t}$$

Describe the pathlines and the streamlines. Note that the pathlines are found by following the motion of each particle, that is, by solving the differential equations

$$dx/dt = u(\vec{r}, t) \text{ and } dy/dt = v(\vec{r}, t),$$

subject to $\vec{r} = \vec{r}_0$ at $t = 0$.
Represent on three different graphics, the streamlines at $t = 0$, the streamlines for $t \to \infty$ and the pathlines for particles found initially in the upper right quadrant.

[Solutions: streamlines $y = Ax^{\frac{2+2t}{2+t}}$, pathlines $y = \frac{y_0}{4}(1 + x/x_0)^2$.]
[Important: streamlines and pathlines (or particle trajectories) coincide only when the flow is steady.
Differences between streamlines, streaklines and pathlines are well explained with the examples of the NCFMF.
https://www.youtube.com/watch?v=mdN8OOkx2ko
You can also find here another example with a translating vortex, as a model for a hurricane.
https://atmos.washington.edu/~durrand/animations/vort505/vortanim1.html]

## 2.6 Vorticity in polar coordinates

1) Take a plane polar element of fluid of dimensions $dr$ and $rd\theta$. Evaluate the right-hand side of Stokes'theorem

$$\int\int \vec{\omega} \cdot d\vec{S} = \int \vec{u} \cdot d\vec{s}$$

and thereby show that the expression for vorticity in polar coordinates is

$$\omega_z = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta}\right].$$

Also, find the expression for $\omega_r$ and $\omega_\theta$ in polar coordinates in similar manner.

2) Give the expression of the vorticity for a solid-body rotation

$$u_\theta = \omega_0 r \quad \text{and} \quad u_r = 0$$

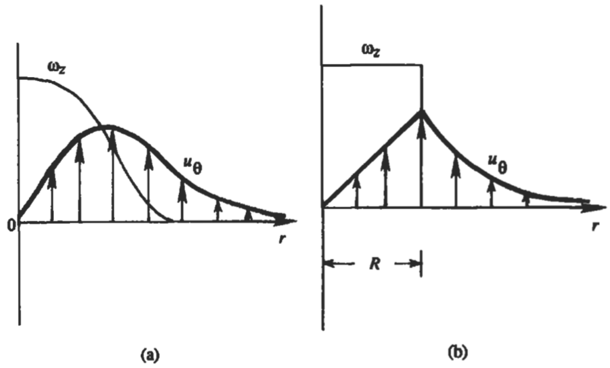


Figure 7: Velocity and vorticity distributions in a real vortex (a) and in a Rankine vortex (b).

## 2.7 Rankine vortex

A Rankine vortex is a 2D model vortex. It is characterized by an uniform vorticity inside a core of radius $R$ and no vorticity outside this core (Fig. 7). Both components $u_r$ and $u_z$ equal zero.
Determine an expression for the velocity components and stream function for a Rankine vortex, assuming that $u_\theta = U$ at $r = R$. For an incompressible and 2D flow in polar coordinates the stream function $\psi$ is given by $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$ and $u_r = -\frac{\partial \psi}{\partial r}$.

[More information about vorticity in general, including vortices
https://www.youtube.com/watch?v=loCLkcYEWD4]

## 2.8 Exercise

Show that the vorticity field for any flow satisfies

$$\nabla \cdot \vec{\omega} = 0$$

## 2.9 Exercise

A flow field on the $xy$-plane has the velocity components

$$u = 3x + y \quad \text{and} \quad v = 2x - 3y$$

Show that the circulation around the circle $(x-1)^2 + (y-6)^2 = 4$ is $4\pi$.

## 2.10 Exercise

Using the indicial notation (and without using any vector identity) show that the acceleration of a fluid particle is given by

$$\vec{a} = \frac{\partial \vec{u}}{\partial t} + \vec{\nabla}\left(\frac{1}{2}\|\vec{u}\|^2\right) + \vec{\omega} \times \vec{u}$$

where $\|\vec{u}\|$ is the magnitude of velocity $\vec{u}$ and $\vec{\omega} = \vec{\nabla} \times \vec{u}$ is the vorticity.

[Hint: we also have $\vec{a} = \frac{\partial \vec{u}}{\partial t} + (\vec{u}.\vec{\nabla})\vec{u}$]

## References

[1] Fluid Mechanics, P. K. Kundu and I. M. Cohen, 4th edition, Academic Press
[2] The National Committee for Fluid Mechanics Films movie series
https://www.youtube.com/playlist?list=PL96364257FFBC4A3A
[3] Hydrodynamique physique, E. Guyon, J.-P. Hulin and L. Petit, EDP SCIENCES
[4] Hydrodynamique physique, M. Fermigier, Dunod

# 1 Momentum transport in viscous flows

## 1.1 Diffusion of momentum

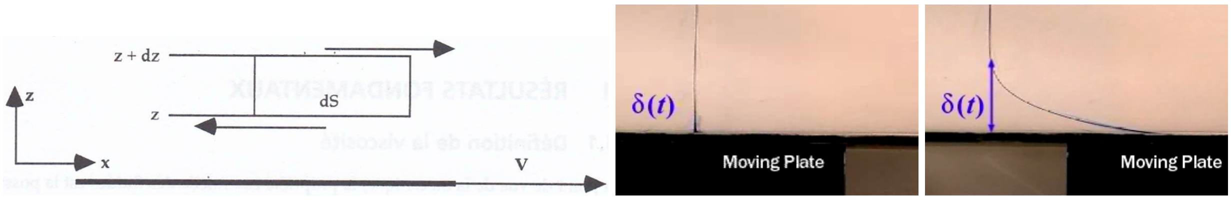


Figure 8: Left: the plate moves at constant velocity $U$ for $t > 0$. Right: two successive snapshots.

A plate found in the plane O$xy$ is in contact with a very large pool of liquid. The liquid viscosity is denoted $\eta$ and its density $\rho$. The plate is initially at rest. At $t = 0$ the plate starts to move in the $x$ direction with a constant velocity $U$. This imposes the boundary conditions $u(z = 0, t) = U$ and $v(z = 0, t) = 0$, with $u$ and $v$ the $x$ and $z$ components of the velocity, .

(1) Assuming that the flow is incompressible, show that the $z$ component of the velocity is equal to 0 and that $\vec{u} = u(z, t)\vec{e}_x$.

(2) Using the two components of the Navier-Stokes equation, show that the pressure is constant and that $u(z, t)$ follows the diffusion-like equation

$$\frac{\partial}{\partial t}u(z,t) = \nu \frac{\partial^2}{\partial z^2}u(z,t) \tag{1}$$

Give the expression and the dimension of $\nu$.

(3) The initial and boundary conditions are $u(z, 0) = 0$, $u(0, t) = U$ and $u(\infty, t) = 0$. From dimensional considerations and analysis show that the velocity $u(z, t)$ can be written

$$u/U = F(z/\sqrt{\nu t}),$$

where $F$ is a dimensionless function.

(4) Equation 1 can be solved using the self-similar variable $\xi = z/2\sqrt{\nu t}$ (the factor 2 is there for algebraic simplification). Write the equation using this new variable and show that the exact solution has the form

$$u(\xi) = A\int_0^{\xi} \exp(-\xi'^2)d\xi' + B$$

(5) Find the constants $A$ and $B$ and plot $\xi$ as a function of $u/U$. We remind you that $\int_0^\infty \exp(-\xi^2)d\xi = \sqrt{\pi}/2$.

(6) As for any diffusive phenomenon, give the order of magnitude of the time needed to initiate the motion of a liquid layer found at a 1 m distance from the plate. Give this typical time for both water and basaltic magma ($\rho = 3000$ kg.m$^{-3}$ and $\eta = 100$ Pa.s).

## 1.2 Oscillating plate

We keep the geometry of the former exercise but the plate is now oscillating with an amplitude $A$. The boundary condition becomes $u(0,t) = A\omega\cos(\omega t)$.

(1) Find the solution of equation 1 using the form $u(z,t) = Ue^{i(kz-\omega t)}$.

(2) Show that the velocity is decaying over a typical length $\delta$. Give the expression of $\delta$ and its value for a frequency equals to 2 Hz and a kinematic viscosity 1000 times higher than the one of water.
What other diffusive phenomena display the same behavior ? Give examples.

[Remark 1: this exercise is particularly well treated and discussed in the Kundu-Cohen]

[Remark 2: the velocity profile is displayed in this video https://www.youtube.com/watch?v=BHBmWcLJlAI]

[Remark 3: this exercise emphasizes that liquids can not propagate shear waves unlike solids. However both liquids and solids can propagate compression waves.]

# 2 Flows at low Reynolds number

## 2.1 Steady flow in a pipe

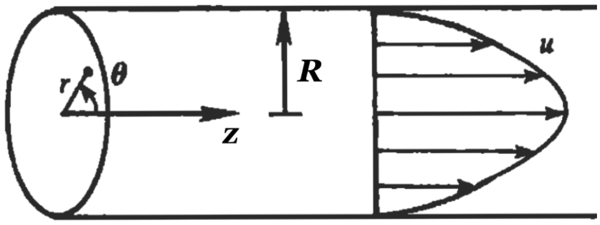


Figure 9: Laminar flow through a tube.

We consider the fully developed laminar flow through a tube of radius $R$. The flow is steady and the velocity components have the cylindrical symmetry and are invariant along $z$ (very long tube approximation).

(1) What drives the flow in this geometry ?

(2) Show that the velocity writes $\vec{u} = u(r)\vec{e}_z$.

(3) Show that the pressure $P$ is a function of $z$ alone and that the pressure gradient is constant and negative. We write in what follows $\frac{\partial P}{\partial z} = -\frac{\Delta P}{L}$ where $L$ is the length of the tube and $\Delta P$ is the (positive) difference of pressure between the entrance and the exit of the tube.

(4) Using the z-component of the momentum equation show that

$$\frac{\eta}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = -\frac{\Delta P}{L}$$

(5) Give the boundary conditions and show that the velocity writes

$$u(r) = \frac{\Delta P}{L}\frac{R^2 - r^2}{4\eta}$$

(6) Show that the flow rate $Q$ is given by

$$Q = \frac{\pi R^4}{8\eta}\frac{\Delta P}{L}$$

This relation is called the Poiseuille law.

[Indications: equations in cylindrical coordinates:]
Incompressible continuity equation:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

($r$,$\theta$,$z$)-components of the Navier-Stokes equation for incompressible flow and without body forces:

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\right) = -\frac{\partial P}{\partial r} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]$$

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right]$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right]$$

## 2.2 Self-propelling liquid trains

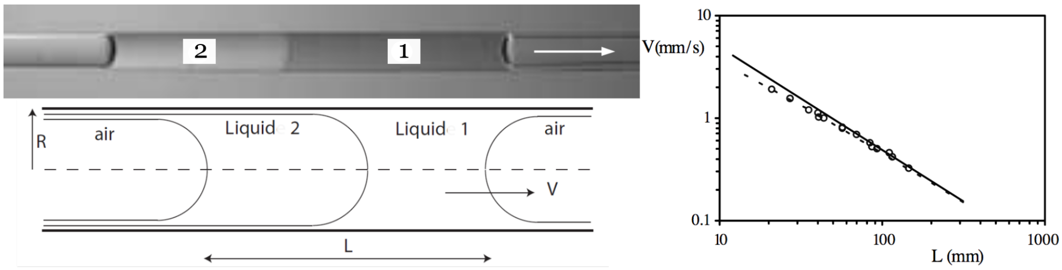


Figure 10: Top: a bislug composed of two wetting liquids spontaneously moves in a tube. Here, the liquids are ethylene glycol and silicone oil, and they are juxtaposed in a glass capillary tube of millimetric diameter. Bottom: representation of a bislug: two interfaces liquid 2/air and liquid 1/liquid 2 replace a liquid 1/air interface. Right: Velocity of a bislug (ethylene glycol/silicone oil with matched viscosities, $\eta = 17$ mPa.s) as a function of its length. The experiment is performed in a glass tube of radius $R = 0.34$ mm prewetted with a film of ethylene glycol. $\gamma_1 - \gamma_{12} - \gamma_2 = 10.0 \pm 0.3$ mN/m for these two liquids. From Bico & Quéré, EPL 2000.

In 1871 Marangoni has shown that it exists couples of liquids inserted in a tube so that this "liquid train" (or "bislug") begins to move and reaches a constant velocity (Fig. 10). The two liquids 1 and 2 have the same viscosity $\eta$ and are juxtaposed in a capillary tube of radius $R$ about a few tenth of a millimeter. Their surface tensions are noted $\gamma_1$ and $\gamma_2$ respectively. The interfacial tension between the two liquids is noted $\gamma_{12}$. Both liquids are wetting the surface of the capillary tube and thin liquid films are deposited, which thicknesses are much smaller than $R$. The length $L$ of the liquid train is much larger than $R$.

(1) By choosing a fixed control volume encompassing the liquid train, show that the driving force is $2\pi R(\gamma_1 - \gamma_{12} - \gamma_2)$.

(2) Experimentally the liquid train moves at constant velocity $V$. Discuss the momentum balance equation for this system and give the other forces if any.

(3) Far from the menisci, the velocity field should be Poiseuille-like. Since $L >> R$ we assume that this hypothesis is valid everywhere in the liquid train. We remind that the Poiseuille law in the cylindrical geometry leads to a parabolic velocity field under the form $\vec{u} = v_{max}(1 - (r/R)^2)\vec{e}_z$, where $v_{max}$ is the velocity at the center of the capillary. Calculate the flow rate $Q$ and the average velocity $V = Q/\pi R^2$ as functions of $v_{max}$ and $R$.

(4) Calculate the components of the viscous stress tensor in the cylindrical coordinates. Deduce the total viscous force applied by the capillary on the liquid train. Express this force as a function of $\eta$, $R$, $L$ and $V$.

(5) Show that the velocity of the liquid train is given by

$$V = \frac{(\gamma_1 - \gamma_{12} - \gamma_2)}{4\eta}\frac{r}{L}$$

Discuss this model with the data obtained experimentally (open circles in Fig. 10).

(6) Plot the pressure profile $P(z)$ inside the liquid train. Is it coherent with both the Poiseuille and the Laplace laws ?

(7) Further developments. Using a dimension analysis, discuss the relevance of all the terms in both the mass balance and Navier-Stokes equations (z-component), including the gravity.
First, justify that the flow is unidirectional (lubrication approximation).
Second, in their article Bico & Quéré observed that a liquid train (R=0.50 mm, L=13 mm, ethylene glycol, $\rho = 1.0\ 10^3$ m$^3$.s$^{-1}$, $\eta = 17$ mPa.s) stops when the tube is tilted with an angle $\alpha = 16°$. Is it surprising ?

[Indications: viscous stress tensor for a newtonian fluid in an incompressible flow]

$$\sigma_{rr} = 2\eta\frac{\partial u_r}{\partial r} \qquad \sigma_{\theta\theta} = 2\eta\left[\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right] \qquad \sigma_{zz} = 2\eta\frac{\partial u_z}{\partial z}$$

$$\sigma_{r\theta} = \eta\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] \qquad \sigma_{\theta z} = \eta\left[\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right] \qquad \sigma_{zr} = \eta\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right]$$

## 2.3 Blood flow in the circulatory system

The blood flow in the circulatory system is complex. Blood is a non-Newtonian fluid, the flow is unsteady and pulsed by the heart, capillaries are deformable, ... . However some features can be analyzed with a simple model: we represent the circulatory system by a succession of rigid tubes subdividing in cascade, from the very large aorta to the smallest capillaries. We subdivide the circulatory systems in 5 levels. Each one is characterized by the number $n$ of blood vessels, the average diameter $d$, the total cross section $S = n\pi d^2/4$, the total volume $V = SL$ and the length of one single vessel $L$ (Tab. 1). At the exit of the heart, the blood

| Level | aorta | large arteries | small arteries | arterioles | capillaries |
|---|---|---|---|---|---|
| $n$ | 1 | 40 | 7100 | $1.6 \times 10^8$ | $5.5 \times 10^9$ |
| $d$ (cm) | 2.6 | 0.8 | 0.06 | 0.002 | 0.0009 |
| $S$ (cm$^2$) | 5.3 | 20 | 20 | 500 | 3500 |
| $V$ (cm$^3$) | 180 | 250 | 250 | 125 | 300 |
| $L$ (cm) | 34 | 12.5 | 12.5 | 0.25 | 0.086 |
| $\Delta p$ (Pa) | | | | | |

Table 1: Features of the different levels of the circulatory system.

flow is unsteady with a pulsation frequency $f = 60$ pulses per minute. The kinematic viscosity of the blood equal 3 mm$^2$.s$^{-1}$ and we take its density as $\rho = 1$ g.cm$^{-3}$.

(1) What is the order of magnitude of the distance $\delta$ over which momentum is diffusing during the period of one heart beat ?

(2) During the cardiac cycle, blood is abruptly ejected inside the aorta. Assuming that the flow remains laminar everywhere in the circulatory system, discuss whether the velocity profile inside the aorta is fully developed (parabolic/Poiseuille-like) or not. If not, at which level of the circulatory system can we consider that the velocity is Poiseuille-like ?

(3) Assuming that the flow is Pouisseuille-like everywhere in the circulatory system, give the expression of the pressure difference $\Delta p$ between the entrance and the exit of each level as a function of the fluid characteristic ($\eta$), the level features ($L$, $S$ and $R$) and the total blood flow rate $Q$.
Fill the last row of Tab. 1.
What is the total pressure difference between the entrance of the aorta and the exit of the capillaries ?
The blood flow rate is $Q = 5$ L.min$^{-1}$. The pressure difference is about 13 kPa (or 100 millimeter of mercury in the unity of physiologists). Compare with your result and discuss.

[Remark: this exercise was proposed by M. Fermigier in “Hydrodynamique physique”, Dunod edition]

[Further developments: boundary layer and fully developed flow in a pipe or between parallel plates (for instance in the chapter “Laminar flow” / book Kundu-Cohen).]

## 2.4 Planar Couette with two immiscible liquids

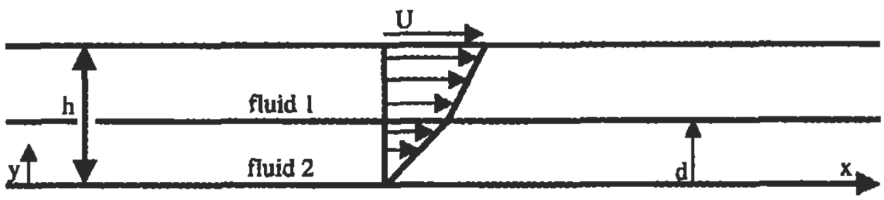


Figure 11: Representation of the problem.

Planar Couette flow is generated by placing a viscous liquid between two infinite parallel plates and moving one plate (say, the upper one) at a velocity $U$ with respect to the other one. The plates are a distance $h$ apart. Two immiscible viscous liquids are now placed between the plates as shown in the diagram.

(1) Solve for the velocity distributions in the two liquids.

## 2.5 Lava flow along a slope

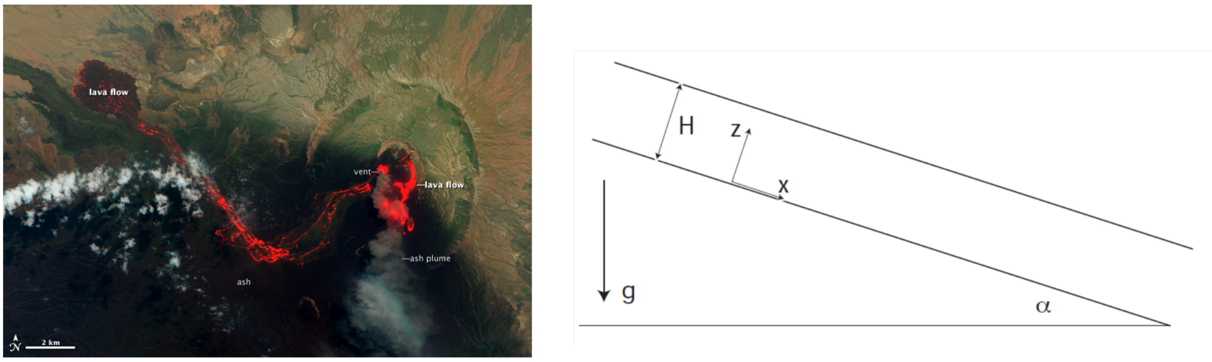


Figure 12: Left: the Nabro eruption imaged by a satellite. Right: model of the lava flow.

We propose to use satellite images to determine the effective viscosity of the lava during the eruption of the Nabro volcano. Images show that the lava flow travelled 12.1 km in 14 days for an elevation change of 555 m. We use a simple 2D model assuming that both the lava thickness $H$ and the slope $\alpha$ can be taken constant (Fig. 12). We assume that the lava length is much larger than the lava height. The flow is steady.

(1) Using the geometry of the model, show that the velocity vector writes $\vec{u} = u(z)\vec{e}_x$.

(2) Project Navier-Stokes equation along the $z$ direction. Explain why the pressure is independent of $x$. Give the expression of the pressure.

(3) Project Navier-Stokes equation along the $x$ direction. Give the two boundary conditions and the expression of $u(z)$.

(4) Calculate the average velocity $\bar{U}$ and demonstrate the Jeffrey's equation that gives the viscosity as a function of the other parameters:

$$\eta = \frac{\rho g \sin(\alpha) H^2}{3\bar{U}}$$

(5) From the satellite images, what are the values of the mean velocity $\bar{U}$ and of the slope $\alpha$ ? Volcanologists suggest to take $H \simeq 5$ m and $\rho = 2900$ kg.m$^{-3}$.
What is the effective viscosity of the lava ?

## 2.6 Honey falling-off from the knife

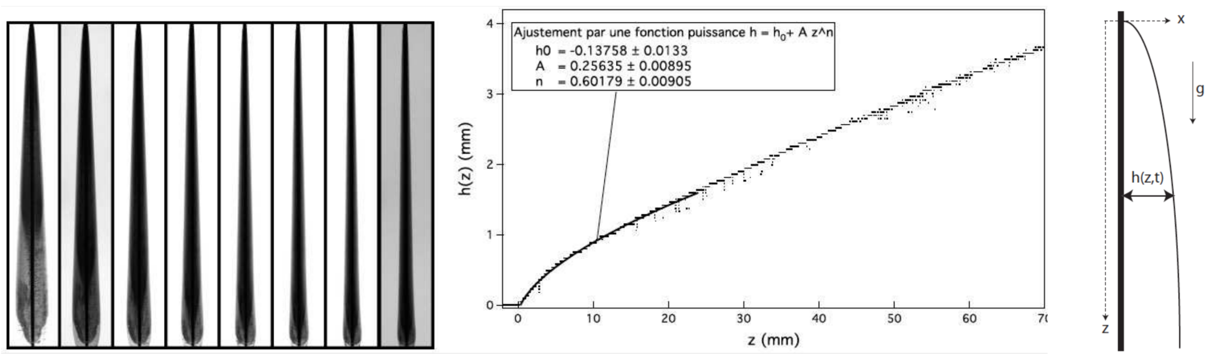


Figure 13: Left: Image sequence of liquid honey pulled from a honey reservoir with a knife (1s between successive images. Total height 10 cm). Middle: height profile of the honey layer $h$ at the beginning of the drainage. Experimental points are interpolated by a power law. Right: Representation of the problem.

It is quite difficult to lift water with a knife, but we can experience it with honey every morning. Once pulled out of the reservoir its height profile displays a particular shape (Fig. 13) far from the tip of the knife. We will model this flow considering that honey is viscous, $\eta = 1$Pa.s. We take $\rho = 1000$kg.m$^{-3}$ for its density.

### 2.6.1 Flow along the knife

We assume that the flow is 2D and we note $h(z,t)$ the thickness profile along the knife.

(1) Justify that the velocity vector $\vec{u}$ is oriented along $z$ only (lubrication approximation). In what follows we note $\vec{u} = u(x,z,t)\vec{e}_z$.

(2) Show that the pressure can be assumed constant everywhere.

(3) Give the Navier-Stokes equation along the z-axis and show that

$$\eta \frac{\partial^2 u}{\partial x^2} = -\rho g$$

Why are the advection terms neglected ?

(4) Using the boundary conditions in $x = 0$ and $x = h(z,t)$, give the velocity profile $u(x,z,t)$. Average $u$ along the $x$-axis to get the average velocity $\bar{U}(z,t)$ along the slice $z$.

(5) Show that the local thickness $h(z,t)$ obeys the equation

$$\frac{\partial h(z,t)}{\partial t} + \frac{\partial h(z,t)\bar{U}(z,t)}{\partial z} = 0$$

(6) The thickness profile look self-affine in space and time. We look for solutions with separated variables: $h(z,t) = f(z).g(t)$.
Calculate $g(t)$ given that the thickness is a decreasing function of time at a given height. Show that $f$ is a power-law function. Compare the exponent with the one measured experimentally (Fig. 13).

[Special focus: the lubrication approximation. This approximation is of paramount importance and is useful to solve numerous problems in fluid mechanics.]

## 2.7 Spin coating

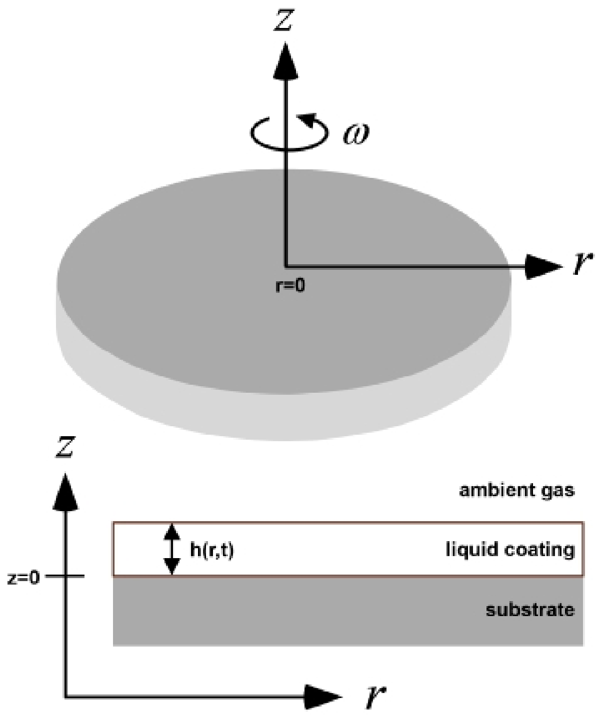


Figure 14: Representation of the problem. We use cylindrical coordinates.

Spin coating consists on the deposition of a liquid on a rotating plate. It is observed that the liquid spreads with time and reaches a constant thickness that evolves as $(\eta/\rho)^{-1/2}\omega^{-1}t^{-1/2}$ with $\eta$ the dynamic viscosity of the liquid, $\rho$ its density, $\omega$ the angular rotation velocity and the time $t$. This process is used to control the thickness of a liquid layer. In this exercise we want to show the formula.

(1) We use the cylindrical geometry. Use the lubrication equation to show that $\vec{v} = v(r, z, t)\vec{e}_r$ and

$$-\eta \frac{\partial^2 v}{\partial z^2} = \rho r \omega^2$$

(2) Using the right boundary conditions in $z = 0$ and $z = h(r, t)$, show that

$$v(r, z, t) = \frac{1}{\eta}\left(-\frac{1}{2}\rho\omega^2 r z^2 + \rho\omega^2 r z h(r, t)\right)$$

(3) Show that the radial flow rate per unit length $q = \int_0^h v dz$ writes

$$q = \frac{\rho\omega^2 r h(r, t)^3}{3\eta}$$

(4) The mass balance equation in this geometry gives $r\frac{\partial h}{\partial t} = -\frac{\partial(rq)}{\partial r}$. Show that $h$ is given by

$$\frac{\partial h}{\partial t} = -K\frac{1}{r}\frac{\partial(r^2 h^3)}{\partial r}$$

with $K = \rho\omega^2/3\eta$.

(5) Show that a uniform thickness function $h(t)$ is solution of the equation. Noting $h(0) = h_0$, show that

$$h(t) = \frac{h_0}{\sqrt{1 + 4Kh_0^2 t}}$$

(6) Give the behavior when $t \to \infty$.

[Remark: this formula is of paramount importance in microfabrication to control the thickness of PDMS layers. The calculations were first made by A. G. Emslie et al., Journal of Applied Physics 29, 858 (1958); https://doi.org/10.1063/1.1723300]

# 3 Conservation laws

## 3.1 Force exerted by a flowing fluid on a pipe bend

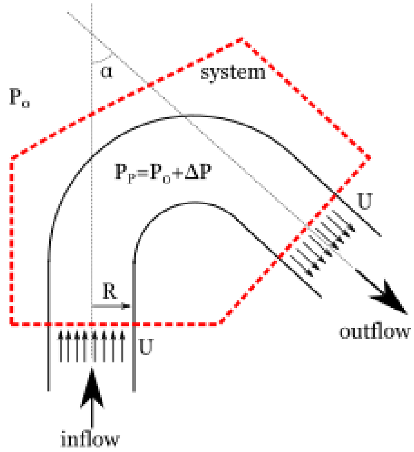


Figure 15: Left: pipe elbow. Right: representation of a pipe bend.

In any flow circulatory system requiring high flow rates, the fluid is pressurized and is carried through pipes. Pipe bends or elbows are used to change the direction of the flow. It is observed that these fittings need good connections since the flowing fluid is exerting a force on them.
We take the geometry of figure 15. The bend makes an angle $\alpha$. The radius of the pipe $R$ is the same before, in and after the bend. We assume that viscous effects can be neglected and that the velocity field inside the straight pipes is the one of a plug flow with a constant velocity $U$. We assume that the flow is steady with a constant flow rate $Q = \pi R^2 U$. We note $P_0$ the atmospheric pressure, $P_p$ the pressure inside the pipe and $\Delta P = P_p - P_0$ the overpressure.

(1) What is the direction of the force exerted on the bend ?

(2) Among the possibilities, chose and argue what is the amplitude of the force that the flow is exerted on the bend. Take the portion delimited by the dashed line in Fig. 15 as the system:

1. $\pi R^2 \rho U^2 \cos(\alpha)$
2. $P_p \pi R^2 * 2\cos(\alpha/2)$
3. $\pi R^2[\Delta P + \cos(\alpha)\rho Q^2/(\pi R^2)^2]$
4. $2\cos(\alpha/2)\pi R^2[\Delta P + \rho Q^2/(\pi R^2)^2)]$
5. $\cos(\alpha)\pi R^2[P_p + \rho Q^2/(\pi R^2)^2)]$
6. other answer

[Special question: think what is the main difference between the force due to the advection of momentum (like rocket propulsion) and the force due to the fluid flow.]

## 3.2 Fire hose

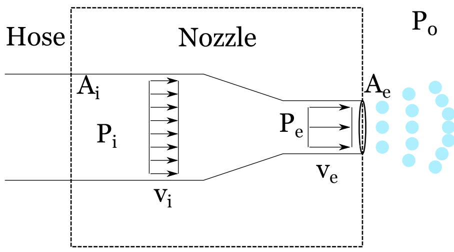


Figure 16: Left: typical position to balance the fire hose reaction. Right: representation of the hose nozzle.

When holding a fire hose, firemen experience the hose reaction. Very often this requires the help of a colleague or special positions: see the movie at https://www.youtube.com/watch?v=6me9iP13zew
In figure 16, the position of the firemen suggests that this force is directed backward (or toward the left). There are a lot of debates about the origin of this reaction. Some people argue that this force is linked to the

constriction inside the hose nozzle (right in Fig. 16). The hose diameter is $d_i = 6.40$ cm. The nozzle diameter is $d_e = 3$ cm. The flow rate is 40 L/s and viscous effects can be neglected.

(1) Calculate the velocities $v_i$ and $v_e$ respectively at the entrance and at the exit of the nozzle. Why can we neglect the viscous effects ?

(2) What are the expressions and values of the pressures $P_i$ and $P_e$ (we note $P_0$ the atmospheric pressure). By performing a momentum balance on the nozzle (dashed system in Fig. 16) give the amplitude and direction of the reaction force experienced by the nozzle.

(3) Is it coherent with the position of the firemen ? Can you identify another source of reaction by looking at the picture ? Discuss this force and the last position showed in the movie.

### 3.3 Flyboard

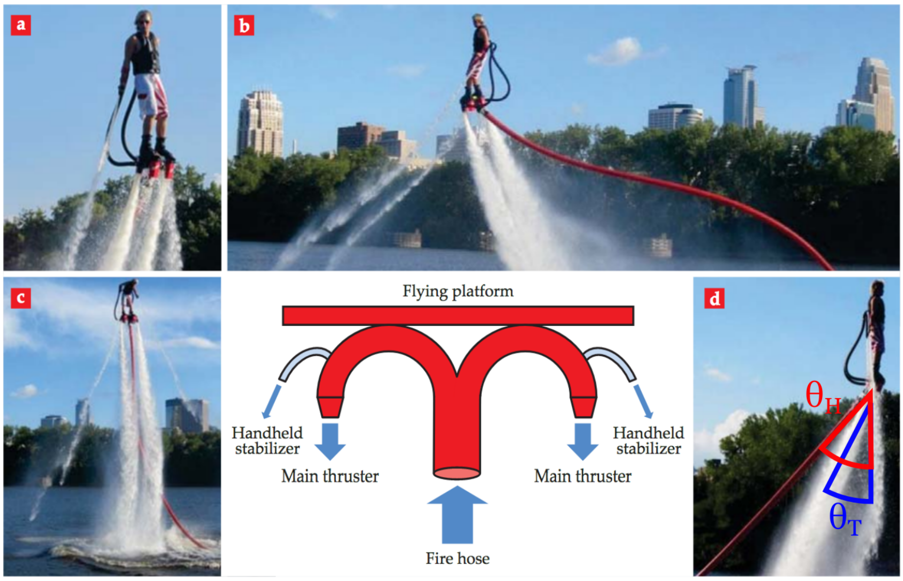


Figure 17: Rider on a flyboard (from Vonk & Bohacek, Physics Today 66, 54 (2013)).

A Flyboard is a hydroflighting device which supplies propulsion to fly in the air or to dive through the water. The rider stands on a board connected by a long hose to a watercraft. Water is forced under pressure to a pair of boots with jet nozzles underneath which provide thrust and to hand stabilizers.
https://en.wikipedia.org/wiki/Flyboard
Fig. 17 gives an example of a rider on a flying platform, which total weight (rider+platform) $W$ equals 910 N. $v_i$ and $v_f$ are the initial (in the hose) and final (expelled) water velocities (in the main thrusters and handheld stabilizers). $A_T = 2.3 * 10^{-3}$ m$^2$ is the cross-sectional area of each of the thrusting nozzles, and $A_S = 0.29 * 10^{-3}$ m$^2$ is the area of each of the two handheld stabilizing nozzles. $A_H = 8.1 * 10^{-3}$ m$^2$ is the cross-sectional area of the fire hose.

(1) Using the mass balance equation, give the expression of $v_i$ as a function of the other parameters aforementioned. In what follows we denote by $\alpha$ the ratio $v_i/v_f$. What is the value of $\alpha$ ?

(2) Using a momentum balance equation, what is the the minimal velocity $v_f$ needed to fly ? We take $\rho = 1000$ kg.m$^{-3}$ for the density of water.

(3) Actually the real velocity is larger that this value. The reason is that in addition to gravity, the hose also pulls down on the craft with a force at least equal to the weight of the hose and its liquid contents. To assess the real velocity the photograph could capture a hovering position from the side. As shown in Fig. 17d the main thrusters and the fire hose tend to angle backward when the jetpack is hovering.
Represent on a scheme the 5 forces (including directions) applied on the jetpack.
By equating the magnitude of the horizontal forces, show that the magnitude $T$ of the tension writes

$$T\sin(\theta_H) = \frac{1}{2}\rho(v_i^2 + v_f^2)A_H\sin(\theta_H) + 2\rho v_f^2 A_T \sin(\theta_T)$$

where $\theta_T$ is the angle the thrusters make with the vertical (in this case about 25°) and $\theta_H$ is the angle the hose makes with the vertical (in this case about 45°).

(4) Using the momentum balance equation along the vertical direction, show that the real value of $v_f$ is 18 m.s$^{-1}$.

### 3.4 Lawn sprinkler

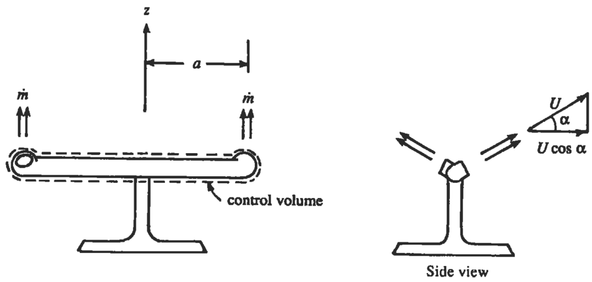


Figure 18: Lawn sprinkler.

Consider a lawn sprinkler as shown in Fig. 18. The area of the nozzle exit is $A$, and the jet velocity is $U$. Find the torque required to hold the rotor stationary.

### 3.5 Impact of a jet on a plane

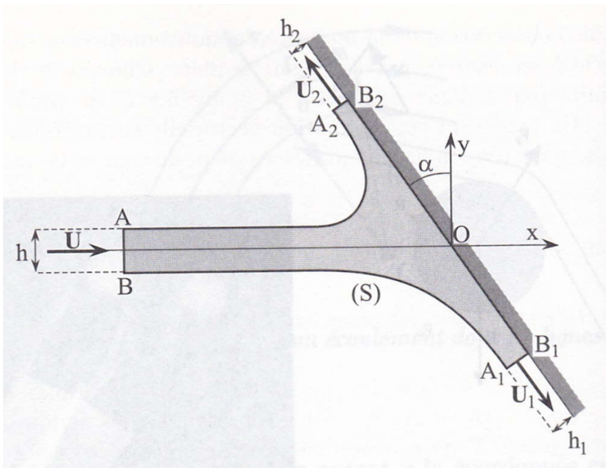


Figure 19: Choice of a stationary volume for a jet impacting a plate.

(1) We consider the two-dimensional problem of a jet impacting a fixed inclined flat plate (fig. 19). Viscous effects are neglected.
Show that $U = U_1 = U_2$ and that $h = h_1 + h_2$.
Using the momentum balance equation on a fixed volume, show that $F_x = \rho U^2(h - (h_1 - h_2)\sin(\alpha))$ and $F_y = \rho U^2(h_1 - h_2)\cos(\alpha)$, where $F_x$ and $F_y$ are the components of the force per unit length exerted on the plate.
$h_1 - h_2$ is still unknown. Assuming that the viscous forces can be neglected, there is no tangential forces to the plate and only the normal force matters. Give the amplitude of the force exerted on the plate.

(2) Case of a 3D cylindrical jet impacting a plate normally ($\alpha = 0°$). The jet has a radius $R$ and an average velocity $U$. Show that the force exerted on the plate writes $\pi\rho R^2 U^2$.

## 4 Boundary layer

We consider the flow generated by a horizontally moving and infinitely thin plate.

1. Derive the dimensionless Navier-Stokes equations from the dimensional ones, we assume a characteristic velocity $U$, and length $L$.
2. What are the boundary conditions for the fluid at the plate, and at infinity?
3. Show that near the plate the Navier Equations are reduced to those of very viscous fluid.
4. Show that far away from the plate, the Navier equations are reduced to the Euler Equations.
5. How would you define the boundary layer thickness.
6. By balancing inertia and viscous shear, predict the functional dependance of the boundary layer thickness.

# 5 Stream function

We consider irrotational flows.

1. Recall what it means.
2. Using the definition of the stream function $\psi(x, y)$ show that the iso contours of the function $\psi$ display the trajectories of the particles embedded in the fluid.

# 6 Irrotational flow around a corner

We consider irrotational flows in a 2D corner.

1. Using complex variables, propose an analytic function for the velocity potential, such that the normal velocity of the fluid is null at the planes defining the corner.

# 7 Flow around a sphere

Consider a sphere, with $\psi = 0$ at $r = a$. We want to compute the velocity distribution when the velocity of the fluid is $U$. We assume that at $r \to \infty$, $\psi \to \frac{1}{2}r^2 U \sin^2\theta$

1. Write the boundary conditions for the fluid velocity at the fluid.
2. Assuming $\psi = f(r)\sin^2\theta$, derive the differential equation for $f(r)$.
3. Search the general solution of the equation using $r^\alpha$, and compute the stream function.